Modeling Temporal Morphological Systems via Lattice Dynamical Systems

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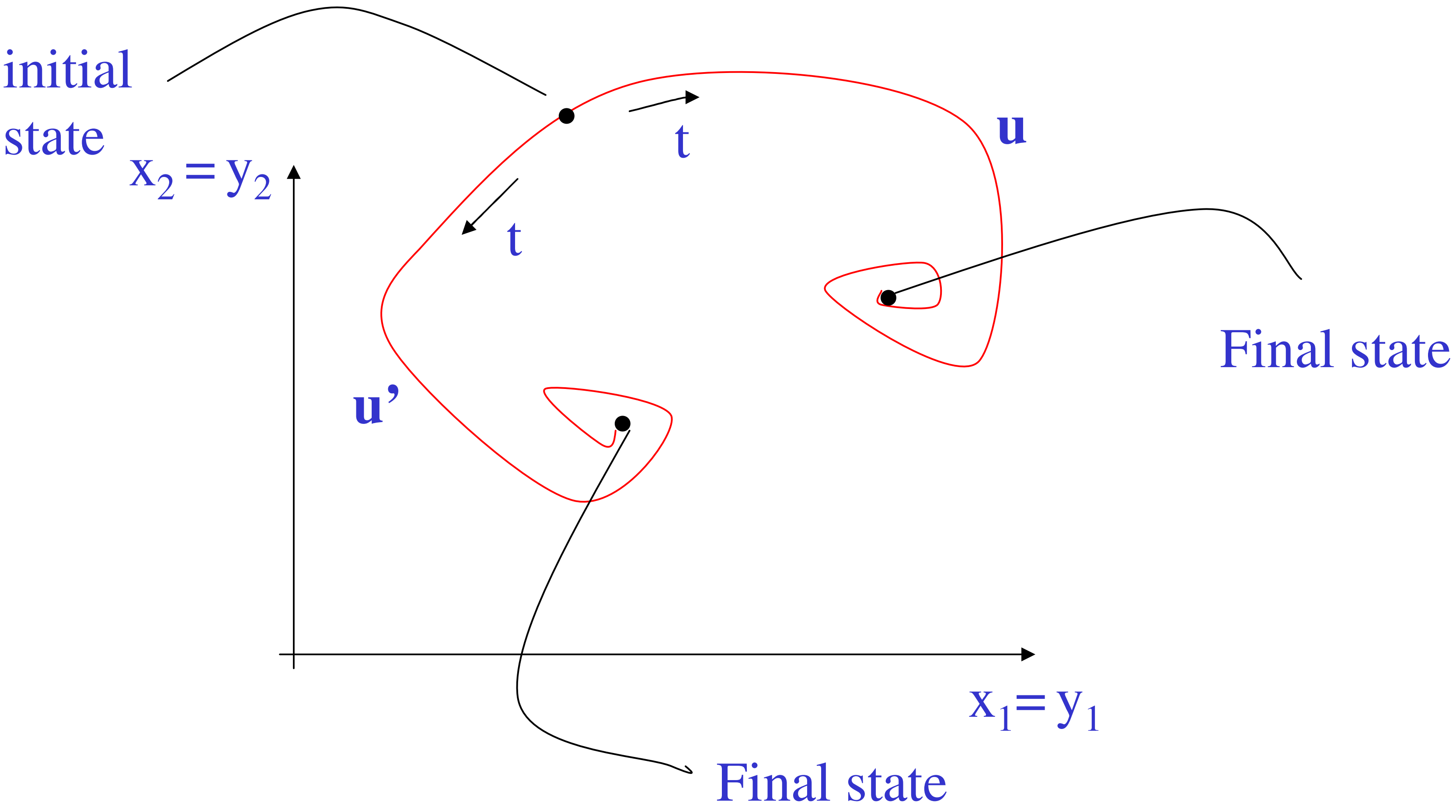
Outline

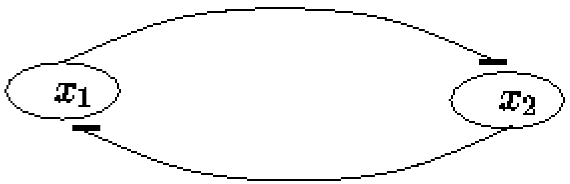
- Introduction
- Lattices
- Operator Representation
- Dynamical Systems
- A model for system identification
- Identification of Boolean Dynamical Systems
- Conclusion

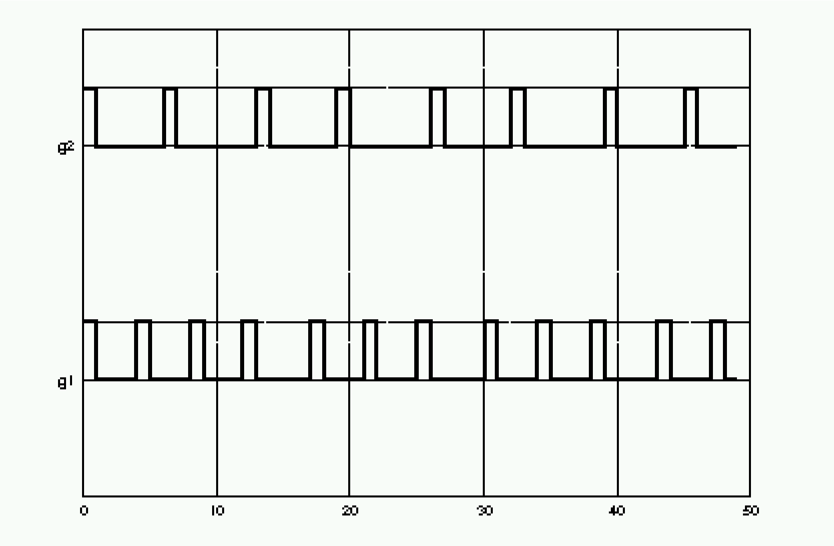
Introduction

Dynamical Systems



Sequential Switching Circuits





$$\phi_1(\mathbf{x}[i-5],\mathbf{x}[i-4],\mathbf{x}[i-3],\mathbf{x}[i-2],\mathbf{x}[i-1],\mathbf{x}[i]) = \overline{x}_1[i-3]\cdot\overline{x}_1[i-2]\cdot\overline{x}_1[i-1]\cdot\overline{x}_2[i-5]\cdot\overline{x}_2[i-3]\cdot\overline{x}_2[i-1]$$

$$\phi_2(\mathbf{x}[i-5],\mathbf{x}[i-4],\mathbf{x}[i-3],\mathbf{x}[i-2],\mathbf{x}[i-1],\mathbf{x}[i]) = \overline{x}_1[i-4]\cdot\overline{x}_2[i-5]\cdot\overline{x}_2[i-4]\cdot\overline{x}_2[i-3]\cdot\overline{x}_2[i-2]\cdot\overline{x}_2[i-1]$$

Mathematical Morphology

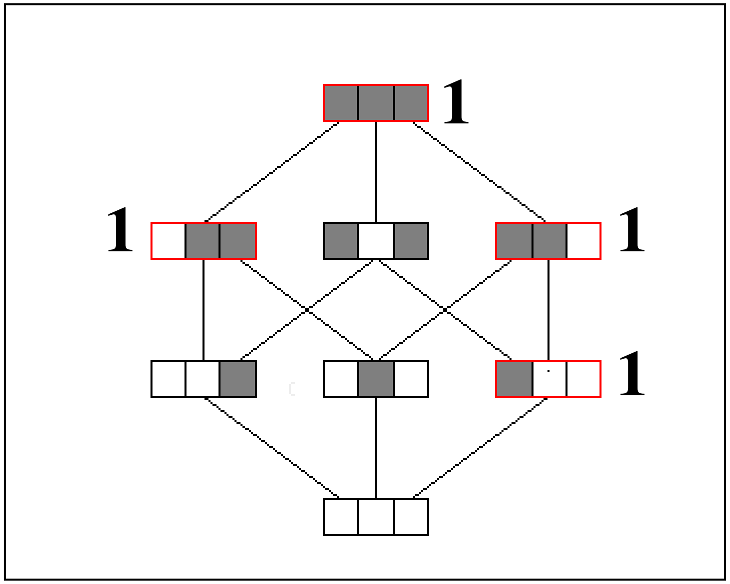
- studies operators between complete lattices, what includes switching functions
- lattice operators are decomposed in terms of simple morphological operators: erosion, dilation, anti-erosion, anti-dilation
- Any lattice operator can be decomposed in a canonical morphological representation

Lattice Dynamical Systems

- We introduce the notion of Finite Lattice Dynamical System (FLDS)
- Give a representation for FLDSs, based on canonical morphological representations
- Develop a theory for statistical identification of FLDSs,

Lattices

Boolean Lattice



Function Lattice

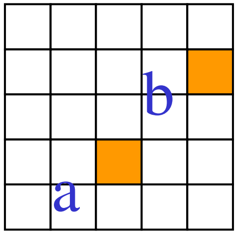
2	0	1	2	2	2
1	0	1	2	2	2
0	-1	1	2	2	2
-1	-1	1	1	1	1
-2	-2	-1	-1	-1	-1
	-2	-1	0	1	2

Operator Representation

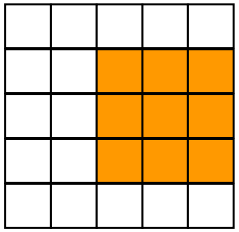
Intervals

- Let a, $b \in$ Fun[W,L], $a \leq b$ iff $a(x) \leq b(x)$, $x \in W$

|W| = 2

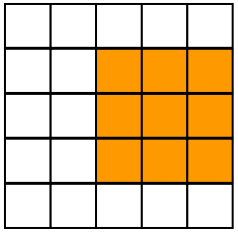


- Interval [a,b] = $\{u \in$ Fun[W,L]: $a \leq u \leq b\}$



Binary Sup-generating

- Sup-generating operator: $\lambda_{a,b}(u)=1 \Leftrightarrow u \in [a,b]$



[a,b]

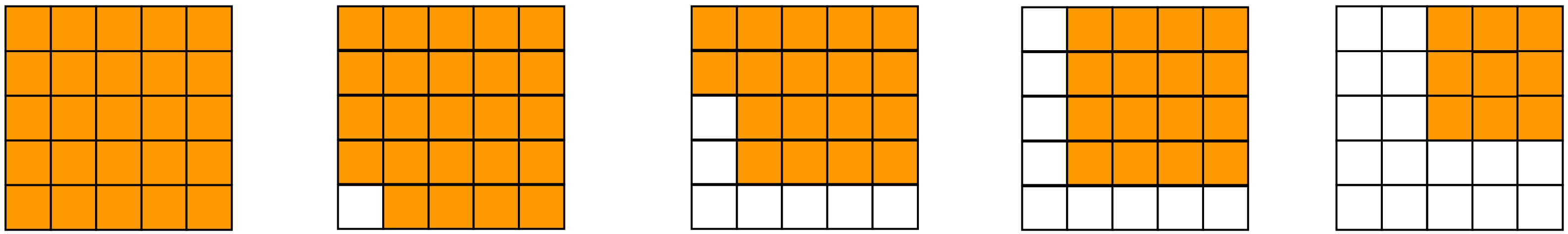
0	0	0	0	0
0	0	1	1	1
0	0	1	1	1
0	0	1	1	1
0	0	0	0	0

$\lambda_{a,b}$

Kernel

Kernel of ψ at *y*: $K(\psi)(y) = \{u \in \mathrm{Fun}[W,L]: y \le \psi(u)\}$

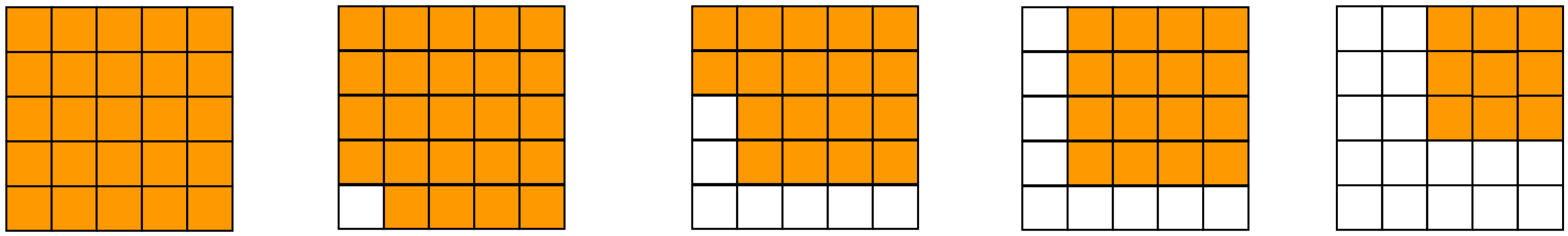
2	0	1	2	2	2
1	0	1	2	2	2
0	-1	1	2	2	2
-1	-1	1	1	1	1
-2	-2	-1	-1	-1	-1
	-2	-1	0	1	2



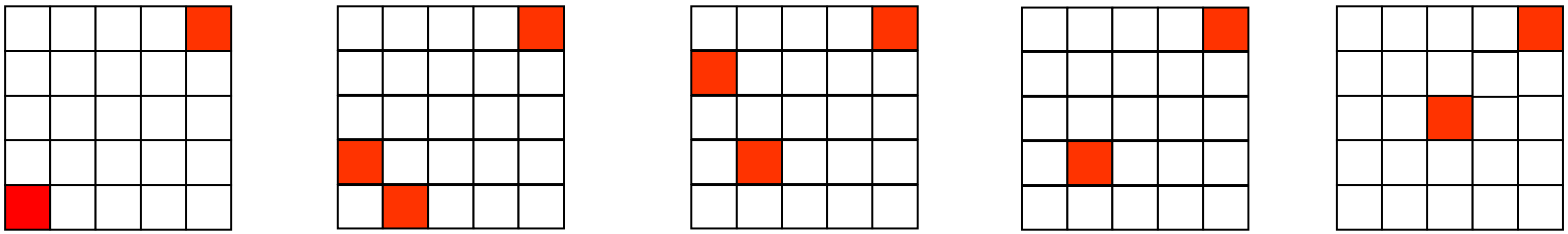
K(ψ)(-2) K(ψ)(-1) K(ψ)(0) K(ψ)(1) K(ψ)(2)

Basis

Basis of ψ at y: $B(\psi)$ is the set of maximal intervals contained in $K(\psi)$



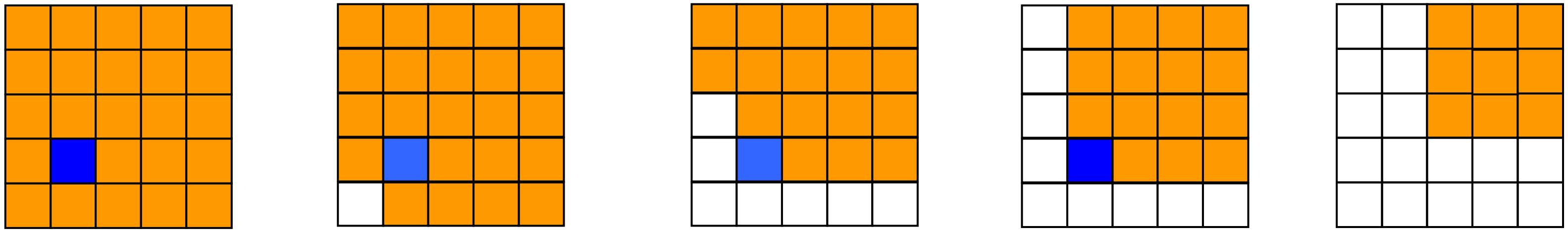
$K(\psi)(-2)$ $K(\psi)(-1)$ $K(\psi)(0)$ $K(\psi)(1)$ $K(\psi)(2)$



$B(\psi)(-2)$ $B(\psi)(-1)$ $B(\psi)(0)$ $B(\psi)(1)$ $B(\psi)(2)$

Canonical Representation

$$\psi(u)=\bigcup\left\{y\in M:\bigcup\left\{\lambda_{a,b}(u):[a,b]\in B(\psi)(y)\right\}=1\right\}$$



K(ψ)(-2) K(ψ)(-1) K(ψ)(0) K(ψ)(1) K(ψ)(2)

ψ(-1,-1) = 1

Dynamical Systems

Finite Lattice Dinamical System

$\mathbf{x} : T \to \mathcal{L}^n$ $\quad$ $\mathbf{y} : T \to \mathcal{L}^m$ $\quad$ $\mathbf{x}[t] \in \mathcal{L}^n$

$\mathbf{u} : T \to \mathcal{L}^n$

$$\mathbf{x}[t+1] = \Phi_t(\mathbf{x}[t-N], ..., \mathbf{x}[t], ..., \mathbf{x}[t+N], \mathbf{u}[t-N], ..., \mathbf{u}[t], ..., \mathbf{u}[t+N])$$

$$\mathbf{y}[t] = \Psi_t(\mathbf{x}[t-N], ..., \mathbf{x}[t], ..., \mathbf{x}[t+N], \mathbf{u}[t-N], ..., \mathbf{u}[t], ..., \mathbf{u}[t+N])$$

$S(\Phi_t, \Psi_t)$

$$\Phi_t : \mathcal{L}^{2(2N+1)n} \to \mathcal{L}^n$$

$$\Psi_t : \mathcal{L}^{2(2N+1)n} \to \mathcal{L}^m$$

Representation

$$\mathbf{x}_j[t+1] = \phi_{t,j}(\mathbf{x}[t-N], \ldots, \mathbf{x}[t], \ldots, \mathbf{x}[t+N], \mathbf{u}[t-N], \ldots, \mathbf{u}[t], \ldots, \mathbf{u}[t+N])$$

$$\mathbf{y}_k[t] = \psi_{t,k}(\mathbf{x}[t-N], \ldots, \mathbf{x}[t], \ldots, \mathbf{x}[t+N], \mathbf{u}[t-N], \ldots, \mathbf{u}[t], \ldots, \mathbf{u}[t+N])$$

$$\mathbf{x}_j[t] \in \mathcal{L}$$

$$\phi_{t,j} : \mathcal{L}^{2(2N+1)n} \to \mathcal{L}$$

$$\psi_{t,k} : \mathcal{L}^{2(2N+1)n} \to \mathcal{L}$$

The component functions have canonical morphological representations

System: input-output

$$\mathbf{y}[t] = S_{(\mathbf{x}[0],\ldots,\mathbf{x}[N],\ldots,\mathbf{x}[2N])}(\Phi_t, \Psi_t)(\mathbf{u}[t-N], \ldots, \mathbf{u}[t], \ldots, \mathbf{u}[t+N])$$

$$\mathbf{y} = S_{(\mathbf{x}[0],\ldots,\mathbf{x}[N],\ldots,\mathbf{x}[2N])}(\Phi_t, \Psi_t)(\mathbf{u})$$

Filter

$\mathbf{u} : T \to \mathcal{L}^n$ $\mathbf{y} : T \to \mathcal{L}^m$

$$\Gamma : \mathcal{L}^{(2N+1)n} \to \mathcal{L}^m$$

$$\mathbf{y}[t] = \Gamma_t(\mathbf{u}[t-N], ..., \mathbf{u}[t], ..., \mathbf{u}[t+N])$$

For example, processing of motion images.

Input-free systems

$$\mathbf{x}[t+1] = \Phi_t(\mathbf{x}[t-N], ..., \mathbf{x}[t], ..., \mathbf{x}[t+N])$$

$$\mathbf{y}[t] = \Psi_t(\mathbf{x}[t-N], ..., \mathbf{x}[t], ..., \mathbf{x}[t+N])$$

Causal systems

$$\mathbf{x}[t+1] = \Phi_t(\mathbf{x}[t-N], ..., \mathbf{x}[t], \mathbf{u}[t-N], ..., \mathbf{u}[t])$$

$$\mathbf{y}[t] = \Psi_t(\mathbf{x}[t-N], ..., \mathbf{x}[t], \mathbf{u}[t-N], ..., \mathbf{u}[t])$$

Time translation invariant systems

$$\Phi : \mathcal{L}^{2(2N+1)n} \rightarrow \mathcal{L}^{n}$$

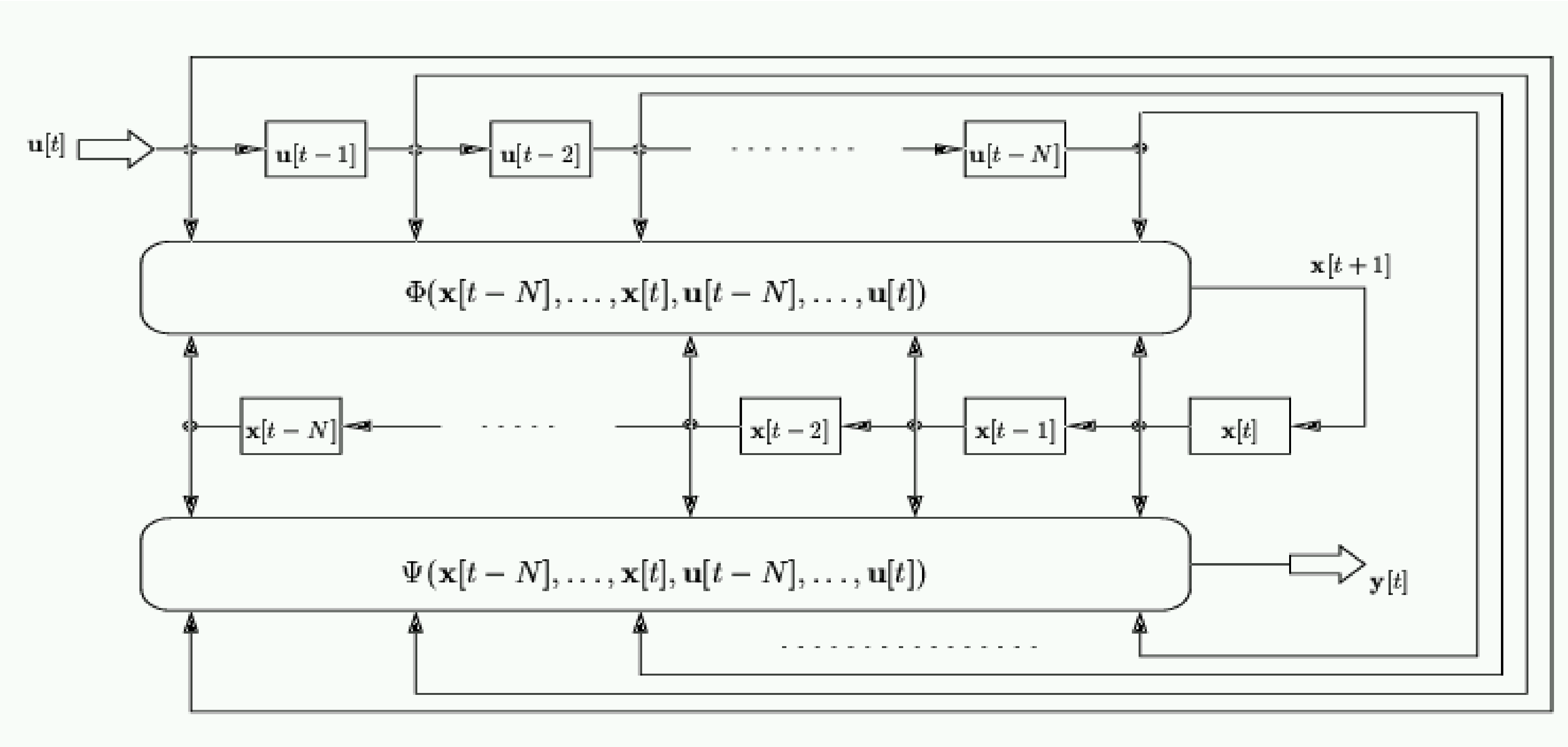
$$\Psi : \mathcal{L}^{2(2N+1)n} \rightarrow \mathcal{L}^{m}$$

$$S(\Phi, \Psi)$$

$$\Phi_t = \Phi$$

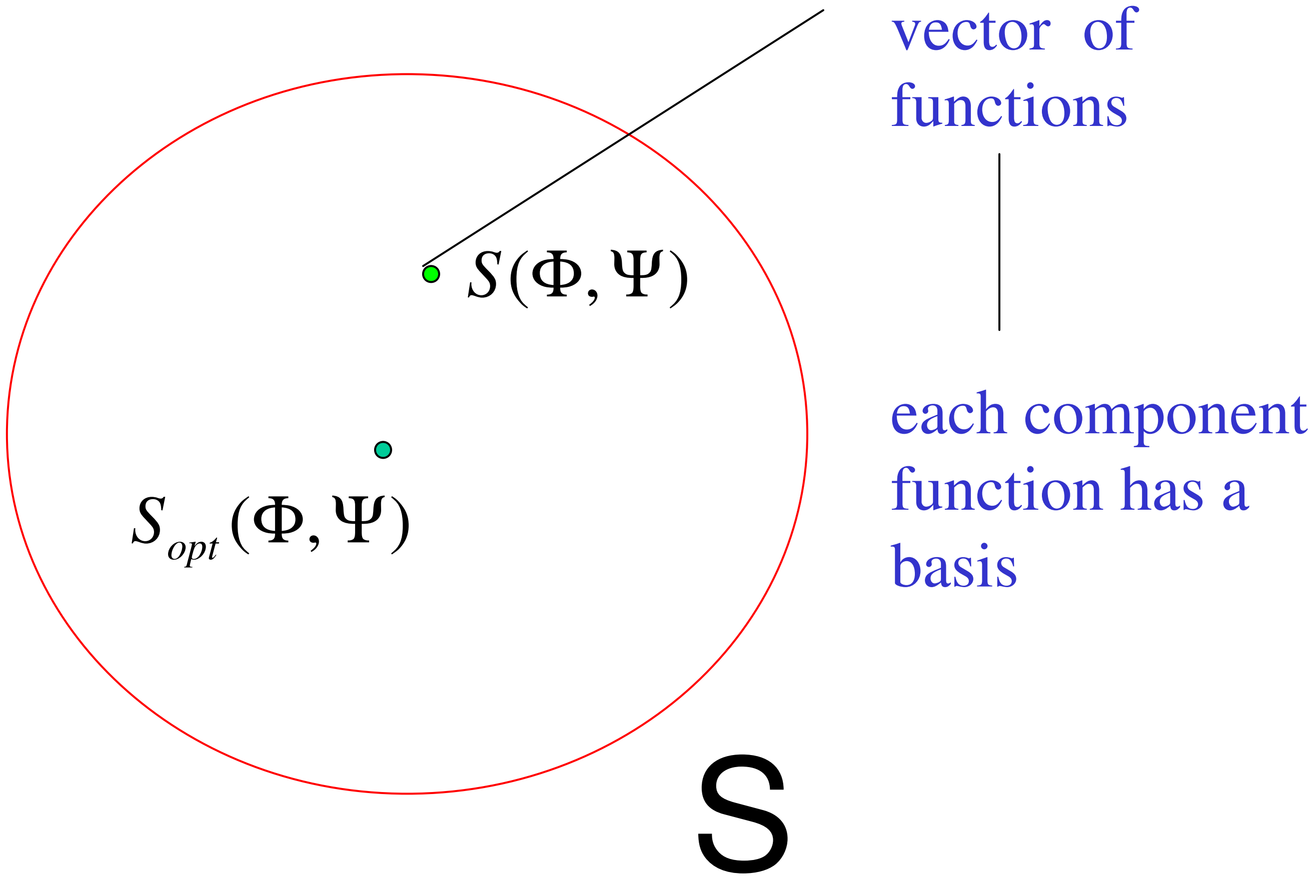
$$\Psi_t = \Psi$$

Causal, time translation invariant



A model for system identification

Model


vector of functions
each component function has a basis
$S(\Phi, \Psi)$
$S_{opt}(\Phi, \Psi)$
S

System Error

$$Er\big(S(\Phi,\Psi)\big) \quad = \quad E\Big[\, l\big(\, S_{(\mathbf{X}[0],\ldots,\mathbf{X}[N])}(\Phi,\Psi)(\mathbf{U}[t-N],\ldots,\mathbf{U}[t-1],\mathbf{U}[t]),\mathbf{I}[t]\,\big)\Big]$$

$$l : \mathcal{L}^m \times \mathcal{L}^m \rightarrow \Re^+$$

Stationary Conditions

$$P(\mathbf{x}[t-N],...,\mathbf{x}[t-1],\mathbf{x}[t],\mathbf{u}[t-N],...,\mathbf{u}[t-1],\mathbf{u}[t],\mathbf{i}[t]) = p$$

$$\begin{aligned} Er\big(S(\Phi,\Psi)\big) &= E\Big[\, l\big(\, S_{(\mathbf{X}[0],...,\mathbf{X}[N])}(\Phi,\Psi)(\mathbf{U}[t-N],...,\mathbf{U}[t-1],\mathbf{U}[t]),\mathbf{I}[t]\,\big)\,\Big] \\ &= \sum_{(\mathbf{x}[t-N],...,\mathbf{x}[t],\mathbf{u}[t-N],...,\mathbf{u}[t],\mathbf{i}[t])\in\mathcal{L}^{2(N+1)n}\times\mathcal{L}^{m}} l\big(\, S_{(\mathbf{x}[t-N],...,\mathbf{x}[t])}(\Phi,\Psi)(\mathbf{u}[t-N],...,\mathbf{u}[t]),\mathbf{i}[t]\,\big)\times \\ &\qquad p(\mathbf{x}[t-N],...,\mathbf{x}[t],\mathbf{u}[t-N],...,\mathbf{u}[t],\mathbf{i}[t]) \end{aligned}$$

Component Error

$$Er_k\left[\,S(\Phi,\Psi)\,\right] = E\Big[\,l_k\big(\,S_{(\mathbf{X}[0],\ldots,\mathbf{X}[N])}(\Phi,\Psi)(\mathbf{U}[t-N],\ldots,\mathbf{U}[t-1],\mathbf{U}[t])_k,\mathbf{I}_k[t]\,\big)\,\Big].$$

$$l_k : \mathcal{L} \times \mathcal{L} \rightarrow \Re^+$$

Additive Loss Function

$l = \sum_{k=1}^{m} c_k l_k \qquad c_k \in \Re^+$

$$Er(S(\Phi, \Psi)) = \sum_{k=1}^{m} Er_k\left[\, S(\Phi, \Psi) \,\right]$$

$e_{MAE}(\mathbf{a}, \mathbf{b}) = \sum_{k=1}^{m} |\, \mathbf{a}_k - \mathbf{b}_k \,|$

$\mathbf{a}, \mathbf{b} \in \{0, 1\}^m$

$e_{MAE} = \sum_{k=1}^{m} e_{k\,MAE}$

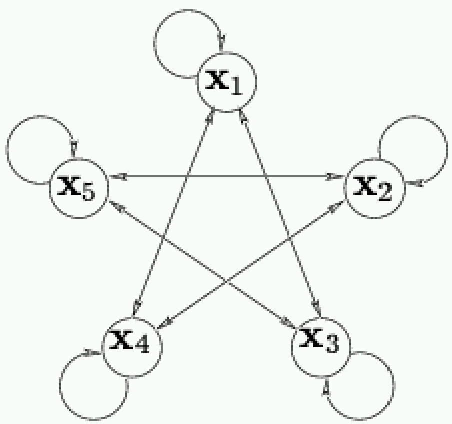
$e_{kMAE}(a, b) = |a - b|$

Independence Condition

- Under additive loss function optimize the system error is equivalent to optimize the system components error
- The problem of system identification is reduced to a family of problems of lattice operator design.

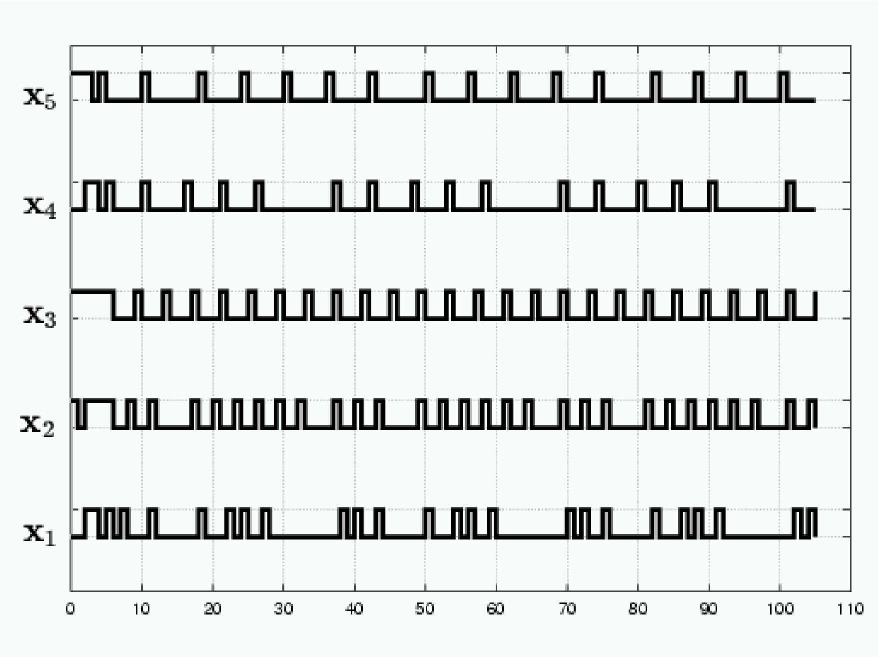
Identification of Boolean Dynamical Systems

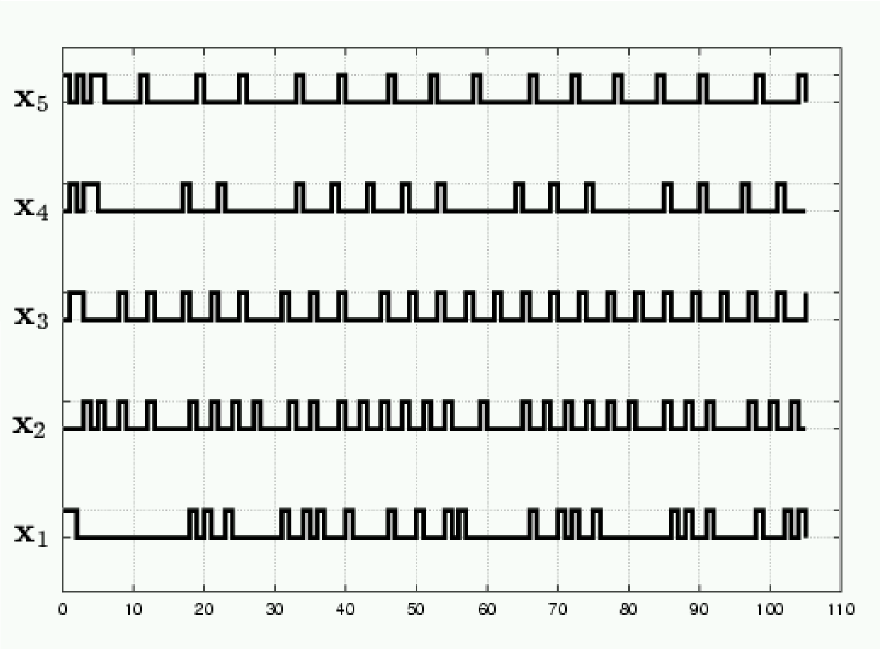
A Boolean System



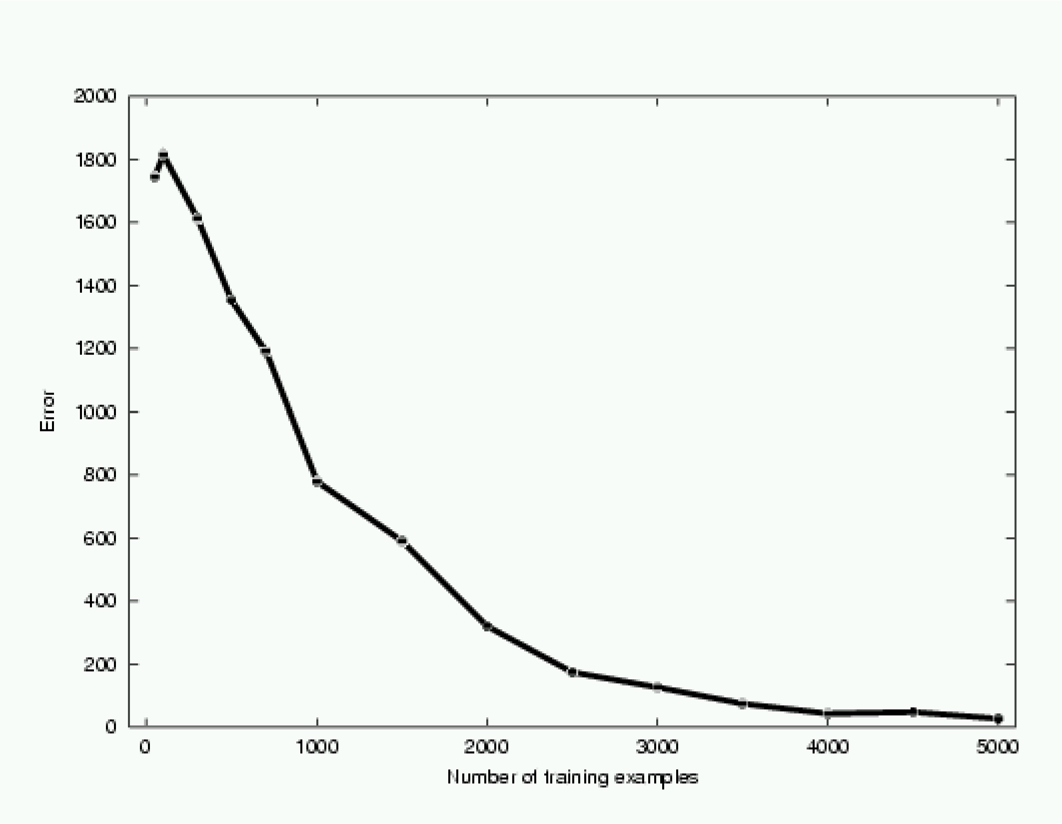
$$
\mathbf{x}_1[t+1] = 1 \Longleftrightarrow \begin{cases}
\mathbf{x}_1[t] = 0 \\
\text{and} \\
\Big[\Big((\mathbf{x}_3[t] = 1 \text{ or } \mathbf{x}_3[t-1] = 1 \text{ or } \mathbf{x}_3[t-2] = 1) \text{ and} \\
\quad (\mathbf{x}_4[t] = 1 \text{ or } \mathbf{x}_4[t-1] = 1 \text{ or } \mathbf{x}_4[t-2] = 1) \Big) \\
\quad \text{or} \\
\quad \Big(\mathbf{x}_3[t] = \mathbf{x}_3[t-1] = \mathbf{x}_3[t-2] = \mathbf{x}_3[t-3] = \mathbf{x}_3[t-4] = 0 \text{ and} \\
\quad \mathbf{x}_4[t] = \mathbf{x}_4[t-1] = \mathbf{x}_4[t-2] = \mathbf{x}_4[t-3] = \mathbf{x}_4[t-4] = 0 \Big) \Big]
\end{cases}
$$

System simulation

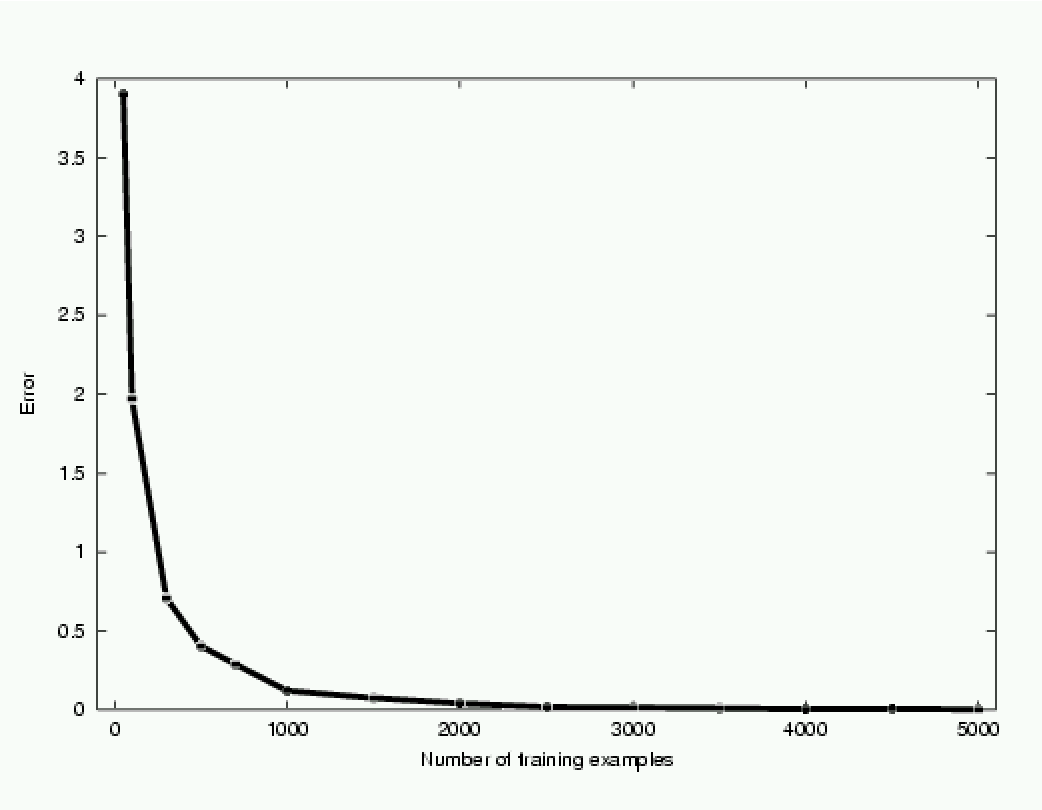


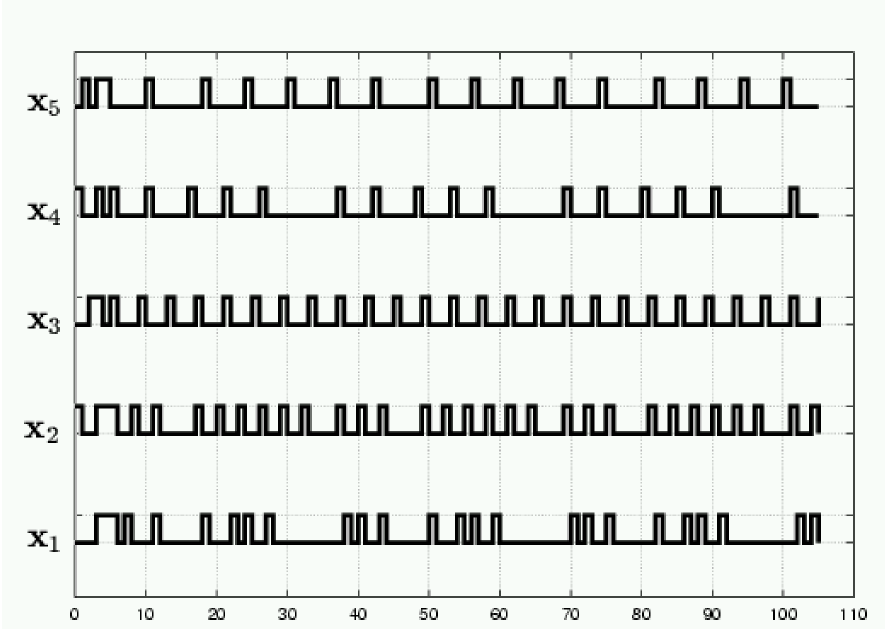


System identification: system error

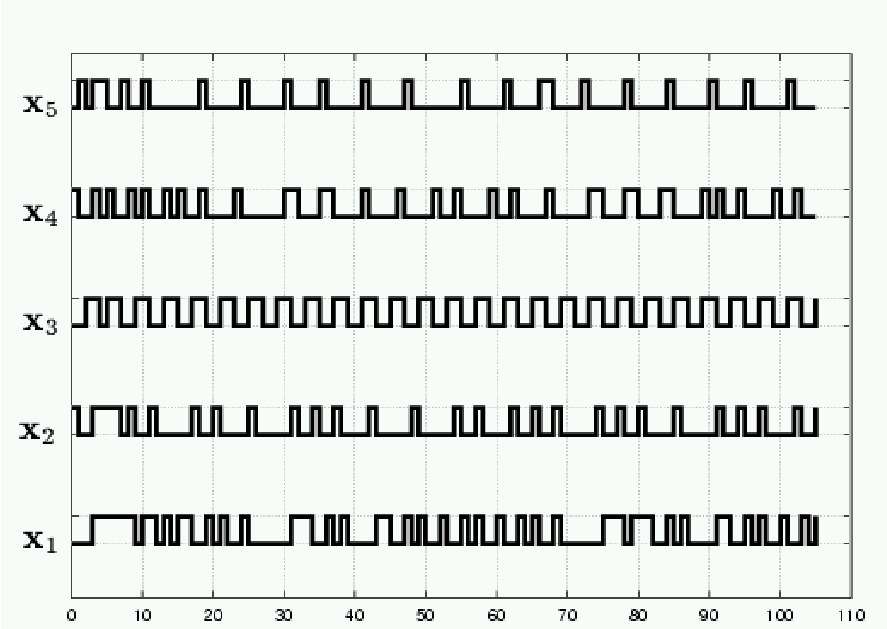


System identification: transition error

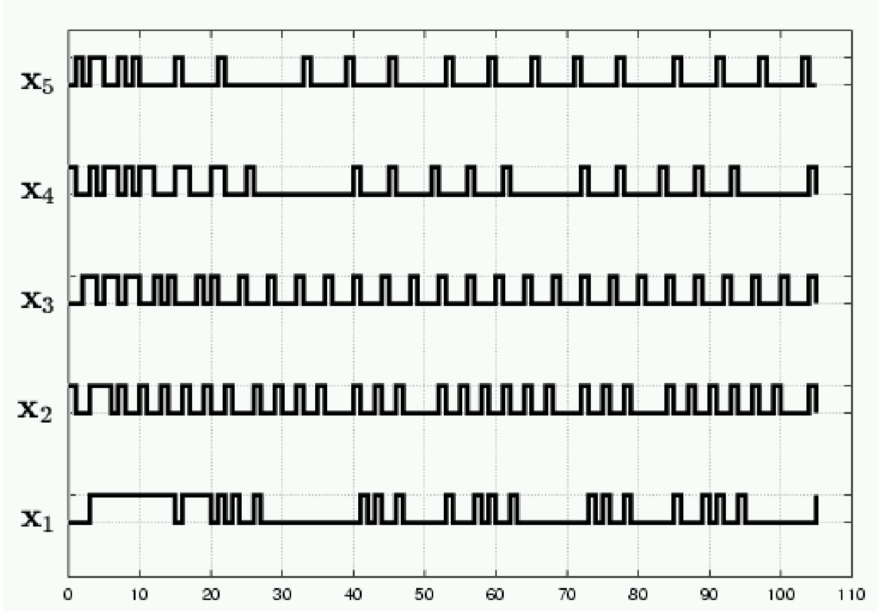




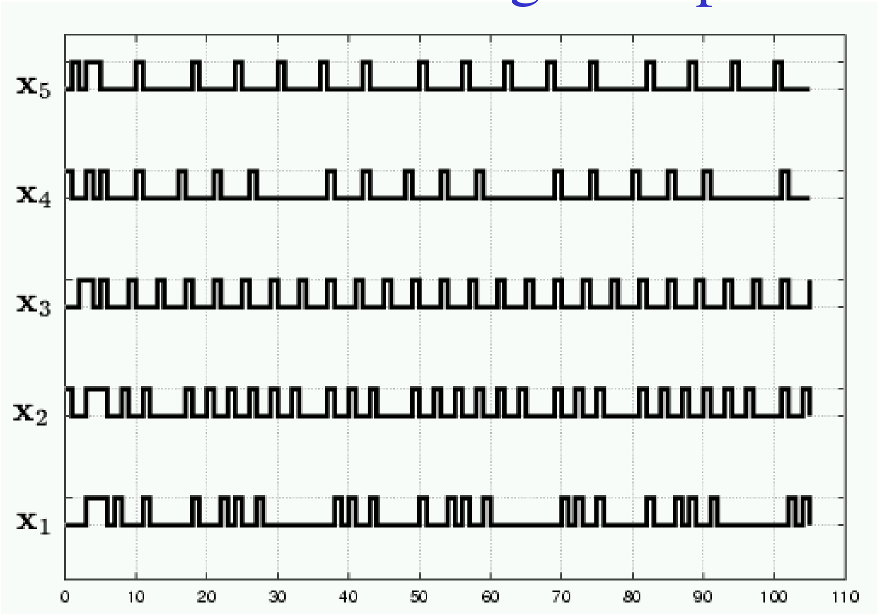
Ideal



Est.: 100 training examples



Est.: 500 training examples



Est.: 1500 training examples

Conclusion

- Introduced the notion of Finite Lattice Dynamical System
- Proposed a model for FLDS identification
- Under additive condition, system identification reduces to a family of problems of lattice operator design
- A Boolean example was presented
- This perspective unifies theories such as switching theory and discrete automatic control