Design of Morphological Operators by Learning

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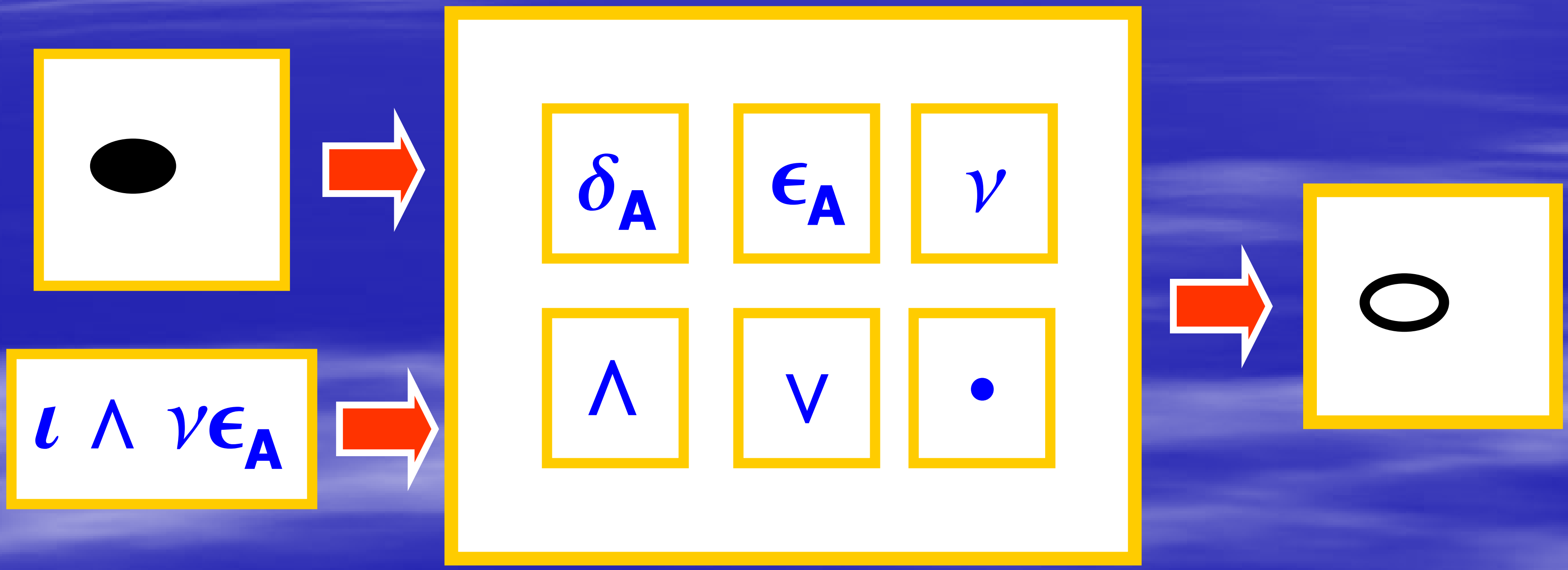
Layout

- Introduction
- Binary operator design: W-operators
- Binary operator design: constraint W-operators
- Gray-scale operator design: apertures
- Gray-scale operator design: stack filters
- Conclusion

Introduction

Morphological Machine (MMach)



Properties

- Any finite lattice operator can be implemented as a program of a MMach
- Finite lattices of practical importance are the lattice of binary and gray-scale images

Morphological Toolbox

- Library of hierarchical functions:
- - primitives: elementary operators and operations;
- - high order operators: primitives and high order operators

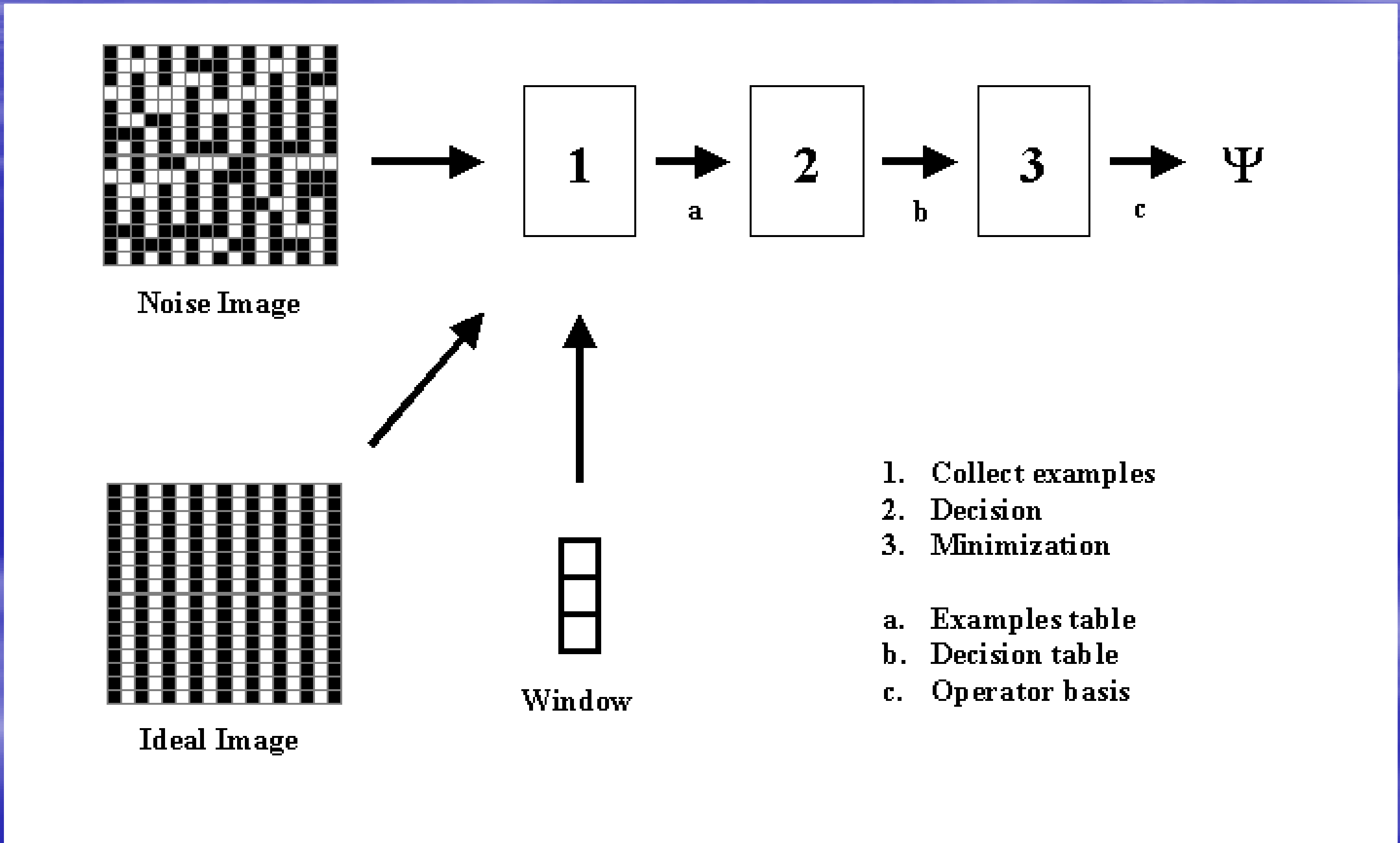
Heuristic Design

- Divide the problem in subproblems
- Each subproblem is solved by a toolbox function
- Integrate the subproblems solution

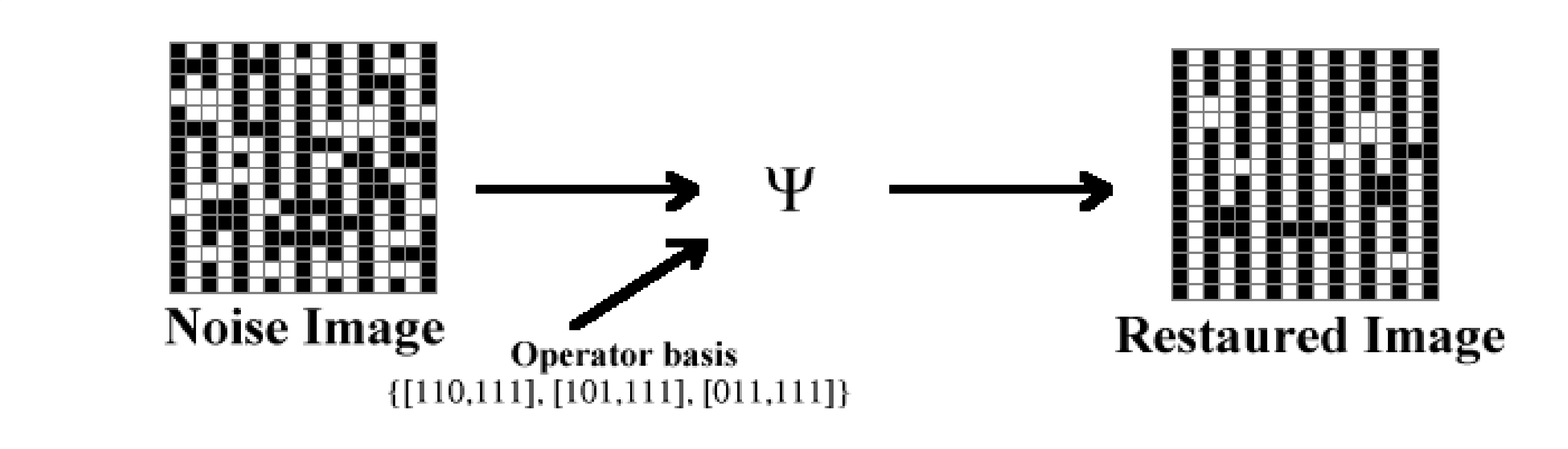
Automatic Design

- Operator learning in a standard representation
- Finding an equivalent and more efficient representation

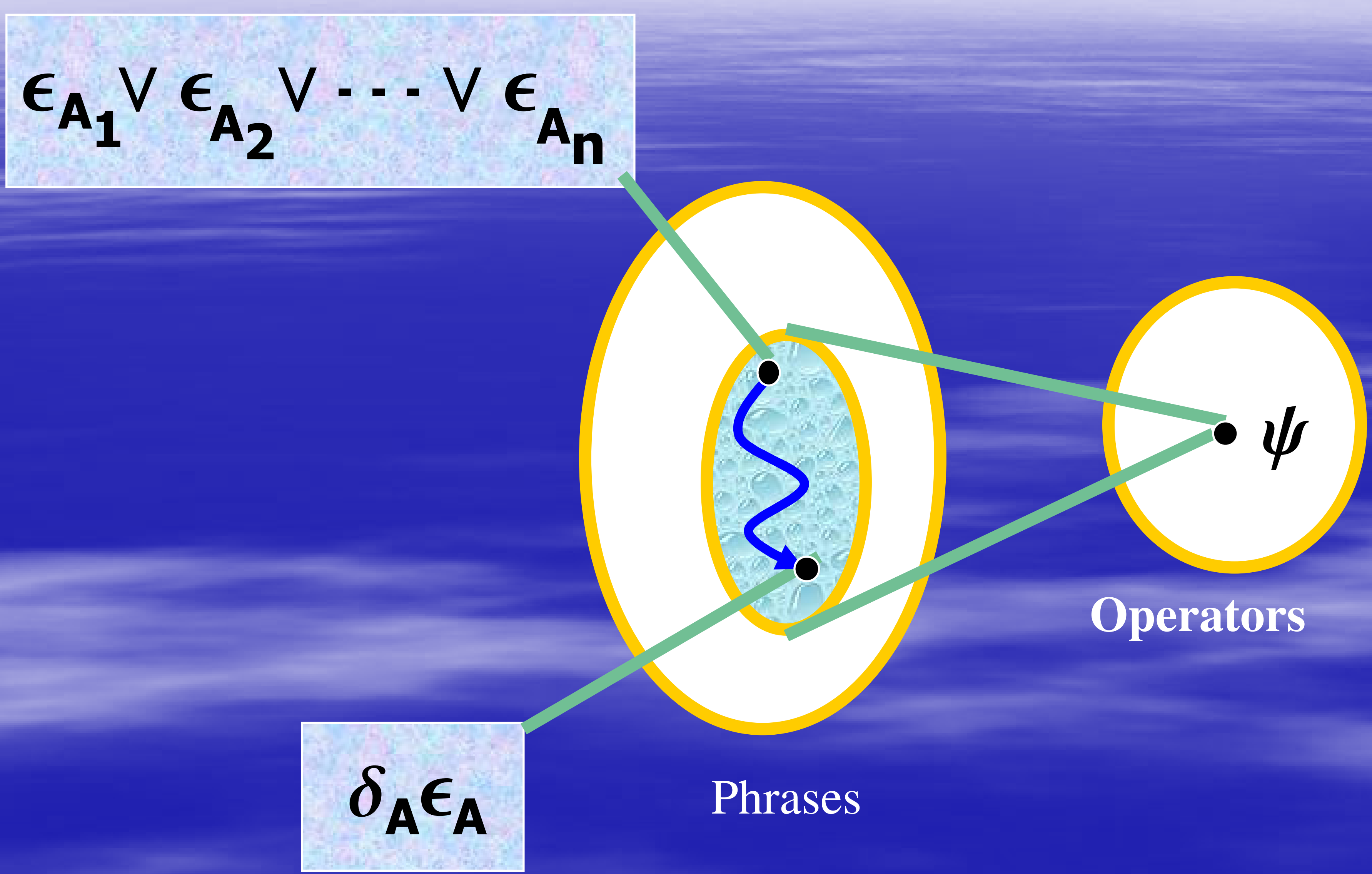
Operator Design



Application

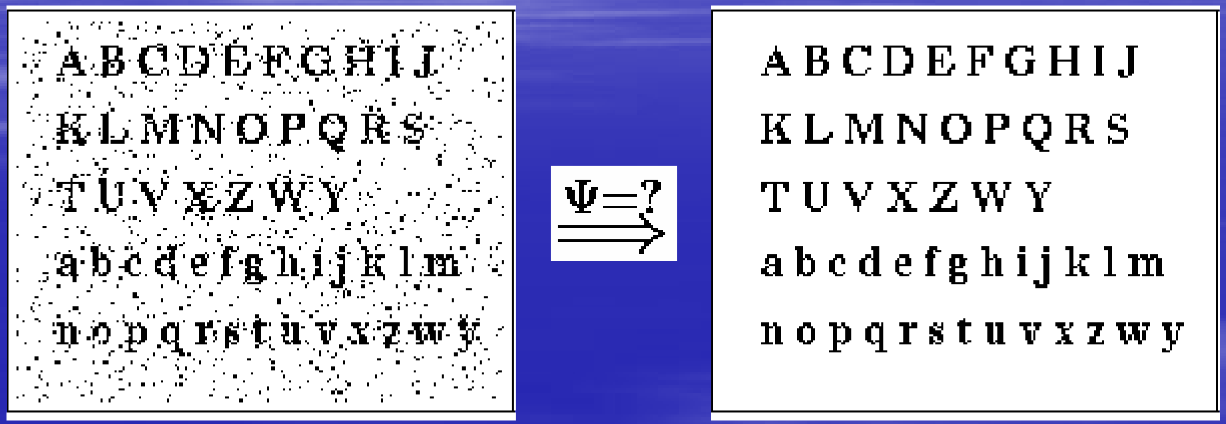


Change of Representation



Operator Design

The problem



observed ideal

Find an image operator that transforms the observed image to the respective ideal (or "close to the ideal") image.

Binary image operators

- Binary image : $f : E \to \{0,1\}$
- Binary images can be understood as sets :

$$f \longleftrightarrow S$$

$$x \in S \Leftrightarrow f(x) = 1 \quad \forall x \in E$$

$(\mathcal{P}(E), \subseteq)$ is a complete Boolean lattice

- Binary image operators = set operators :

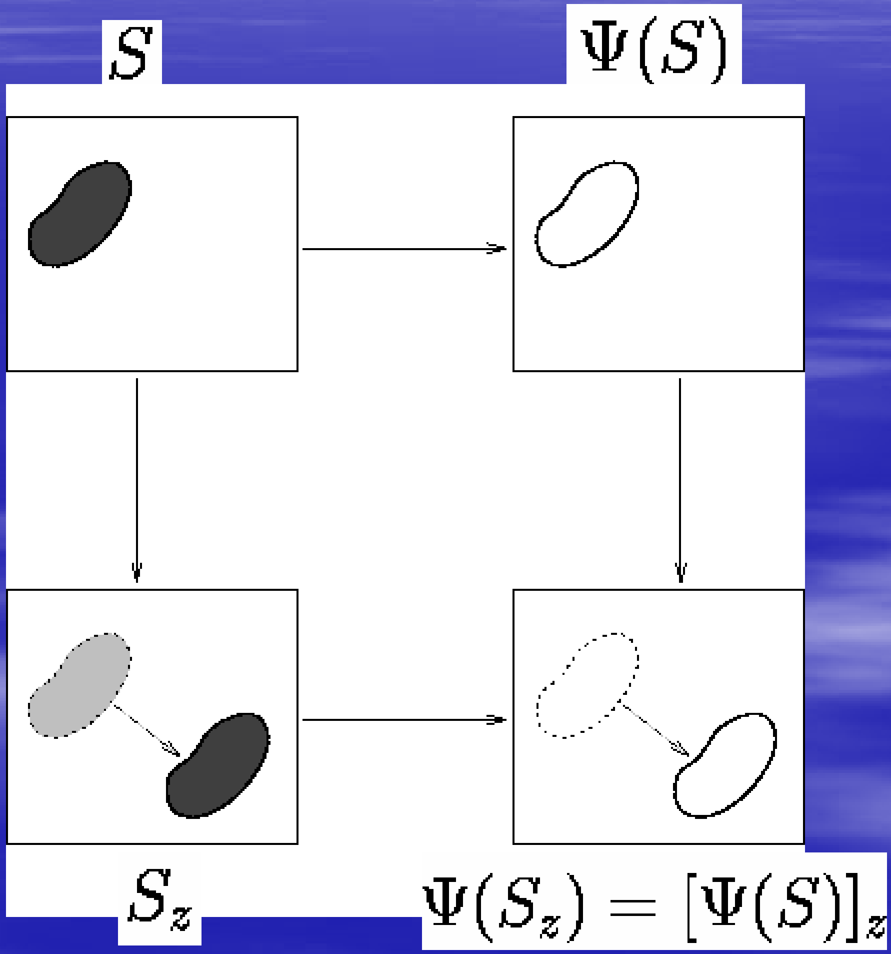
$$\Psi : \mathcal{P}(E) \to \mathcal{P}(E)$$

Translation invariance

- Translation of S by z :

$$S_z = \{x + z : x \in S\}$$

- $\Psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is **translation-invariant** iff $\Psi(S_z) = [\Psi(S)]_z$

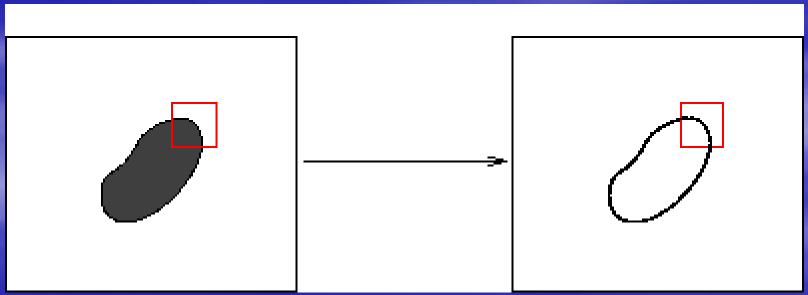


Local definition

Window : $W \subseteq E$

➡ An image operator is locally defined within W iff

$$x \in \Psi(S) \Longleftrightarrow x \in \Psi(S \cap W_x)$$



$S \cap W_z$ $\Psi(S \cap W_z)$

W-operators

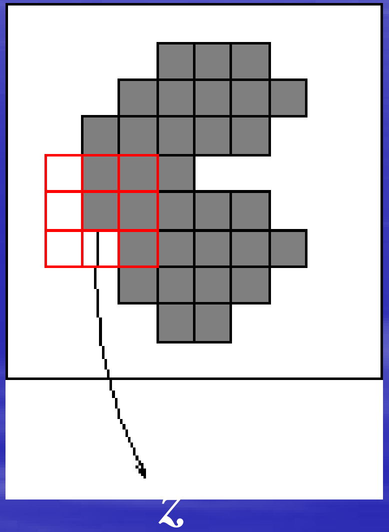
Translation invariance

\+

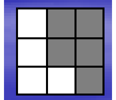
local definition within W

=

W-operators



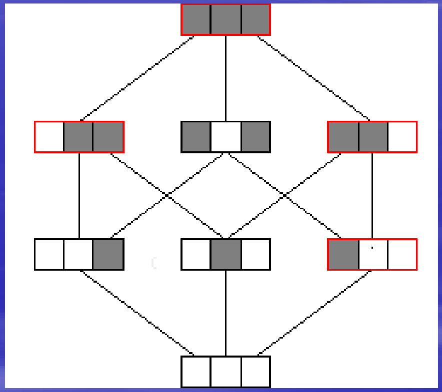
$$\Psi(S)(z) = \psi\big(\quad\big)$$

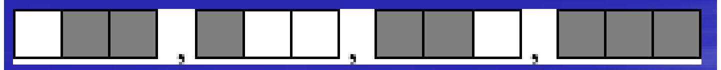


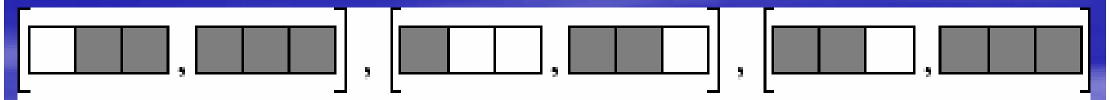
W-operators are characterized by Boolean functions.

Representation

Window $W = 1 \times 3$



$\mathcal{K}(\Psi) = \{$  $\}$

$\mathcal{B}(\Psi) = \{$  $\}$

$$\psi = \lambda_{X11} \cup \lambda_{1X0} \cup \lambda_{11X}$$

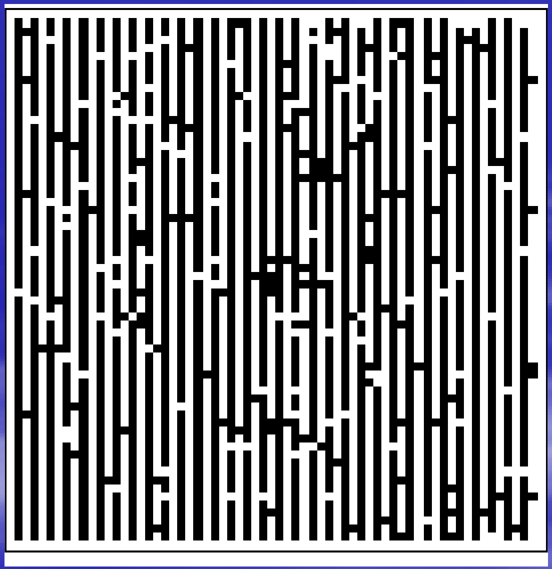
Statistical Hypothesis

X and Y are jointly stationary

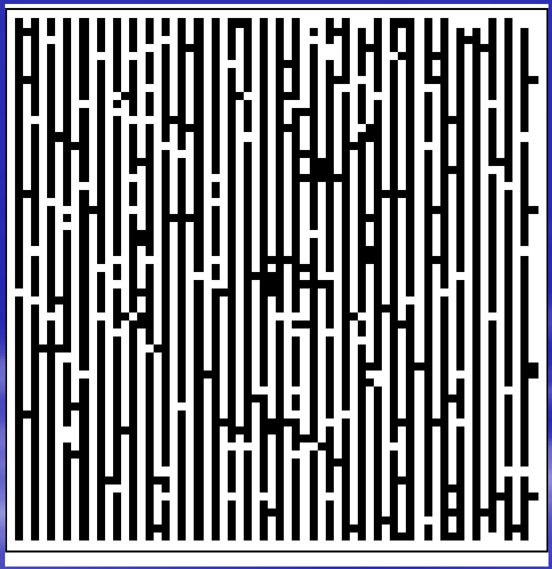
$$P(S \cap W_z, Y)$$

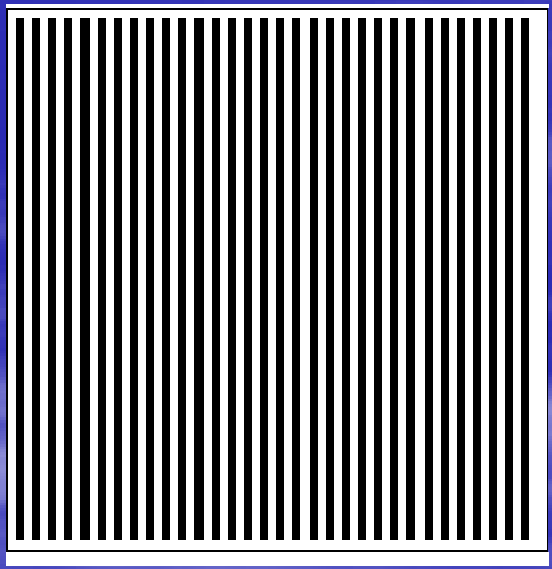
is the same for any z in E

Stationary Process



Join Stationary Process





Error measure

- Design goal is to find a function with minimum risk.

- Risk (expected loss) of a function :

$$R(\psi) = E[l(\psi(X), Y)]$$

X is a random set
Y is a binary random variable

- Loss function

$$l : \{0,1\} \times \{0,1\} \rightarrow R^+$$

MAE example

- *Example* : MAE loss function

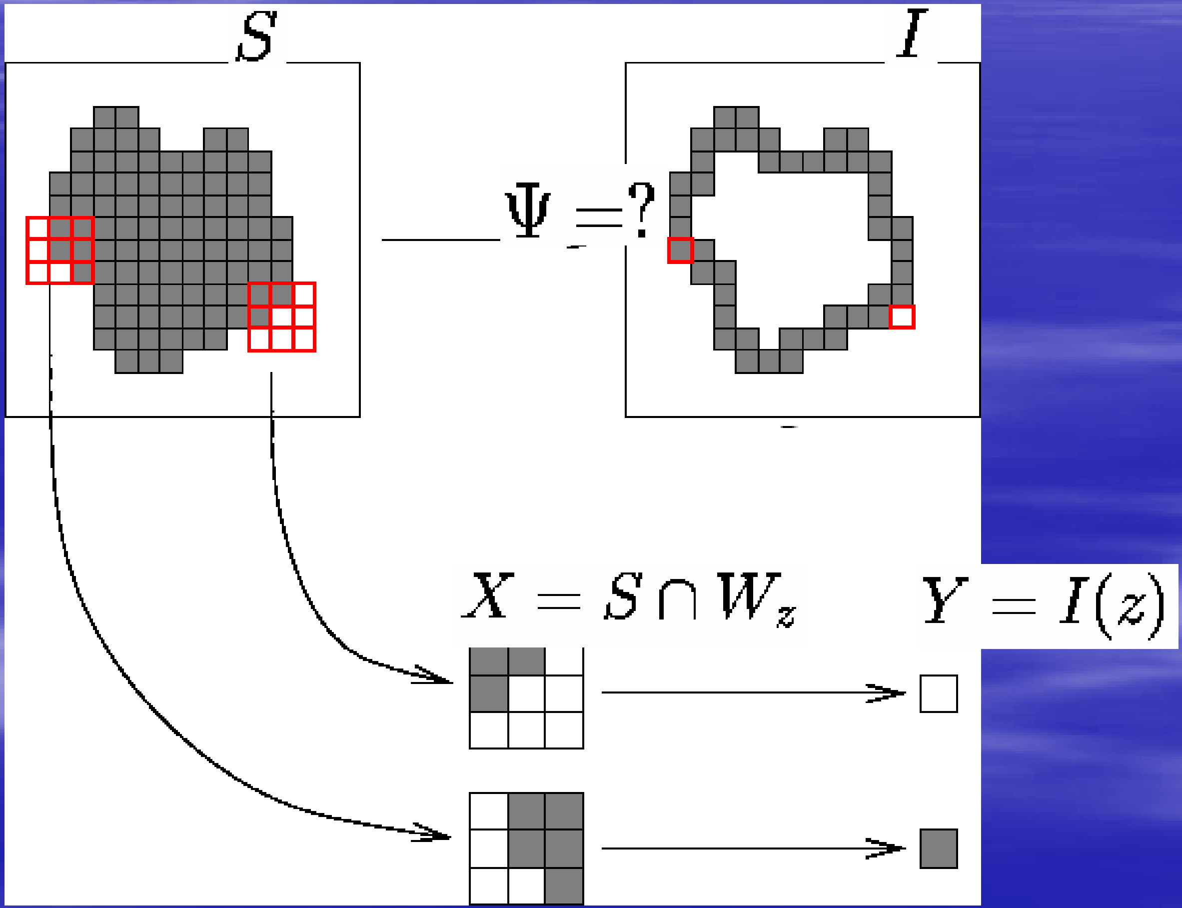
$$l_{MAE}(a,b) = |a-b| \qquad a,b \in \{0,1\}$$

$$MAE\langle\Psi\rangle = E[|\psi(X) - Y|]$$

- Optimal MAE function

$$\psi(X) = \begin{cases} 1 & p(1,X) > p(0,X) \\ 0 & p(1,X) \le p(0,X) \end{cases}$$

Design procedure



PAC learning

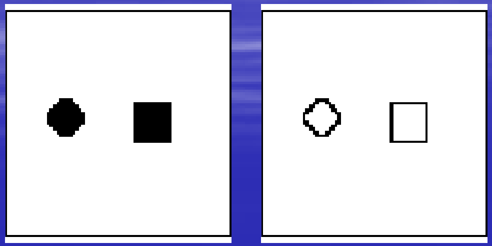
L is Probably Approximately Correct (PAC)

For $m > m(\varepsilon, \delta)$ examples

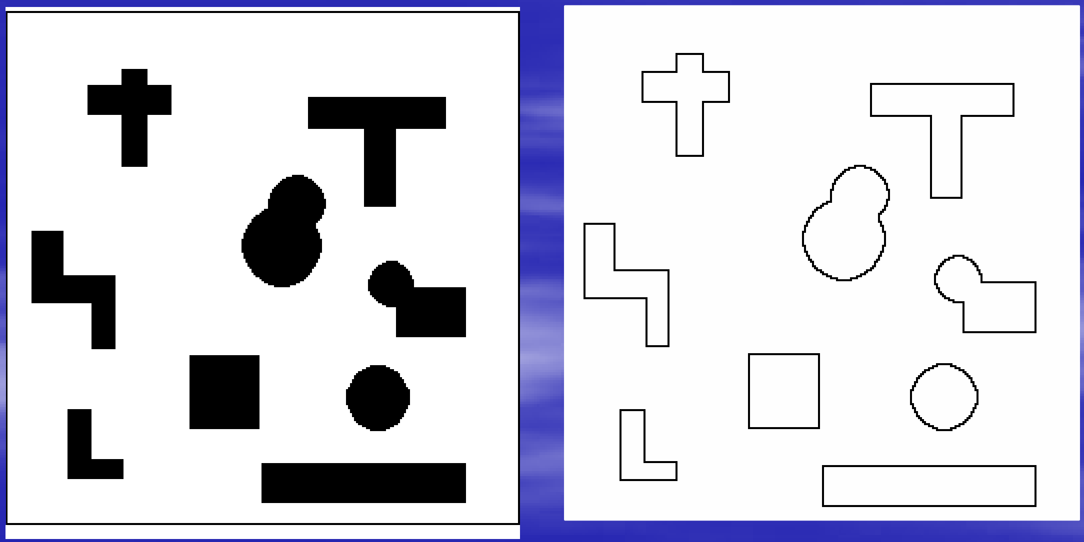
$$\Pr(|R(\psi) - R(\psi_{opt})| < \varepsilon) > 1 - \delta$$

$$\varepsilon, \delta \in (0,1)$$

Edge detection



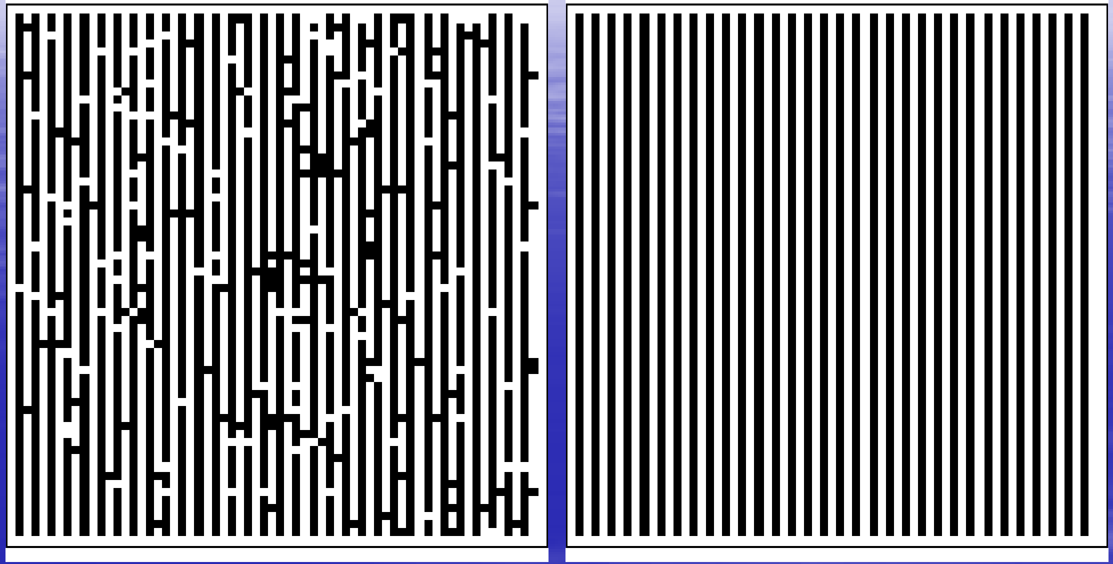
Training images



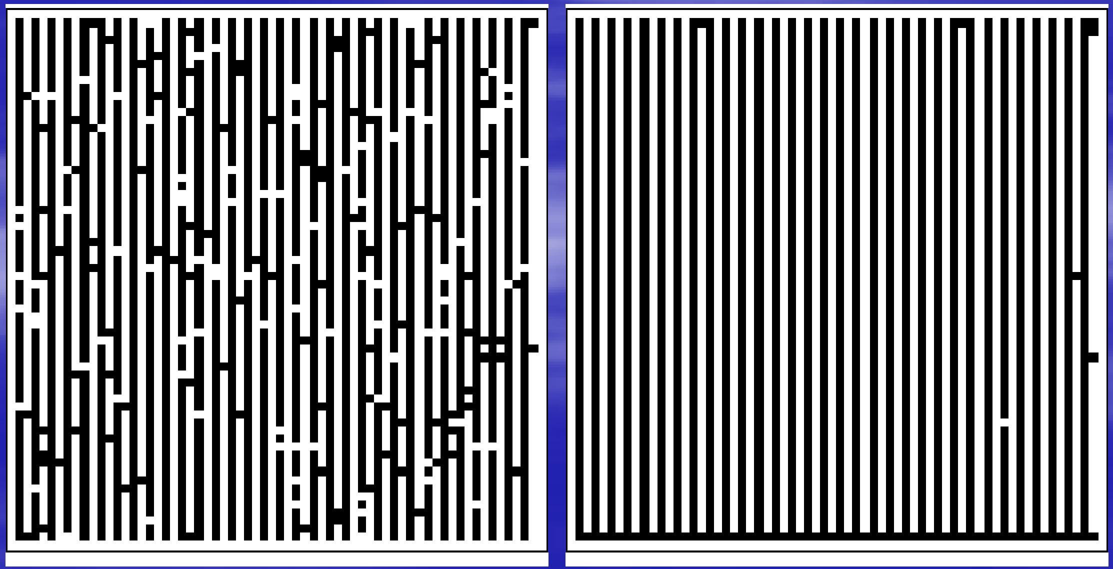
Test images

Noise filtering

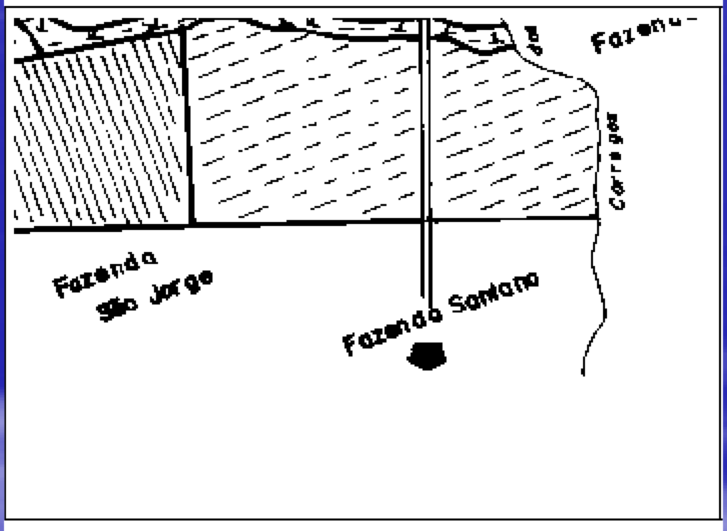
Training images

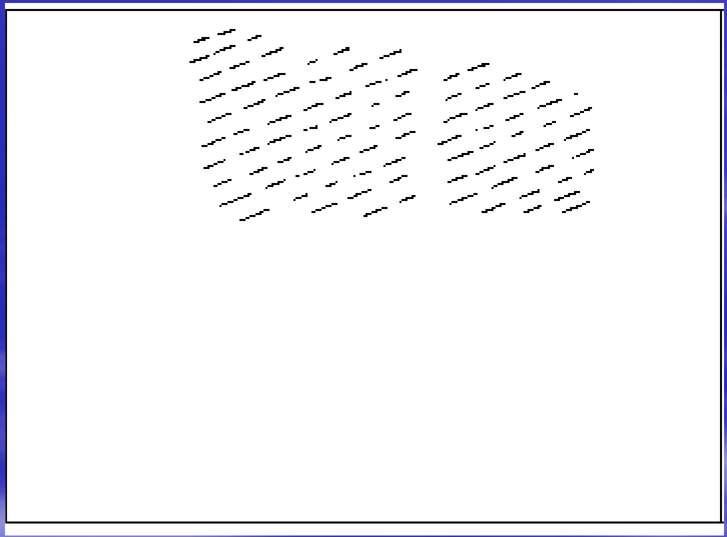


Test images



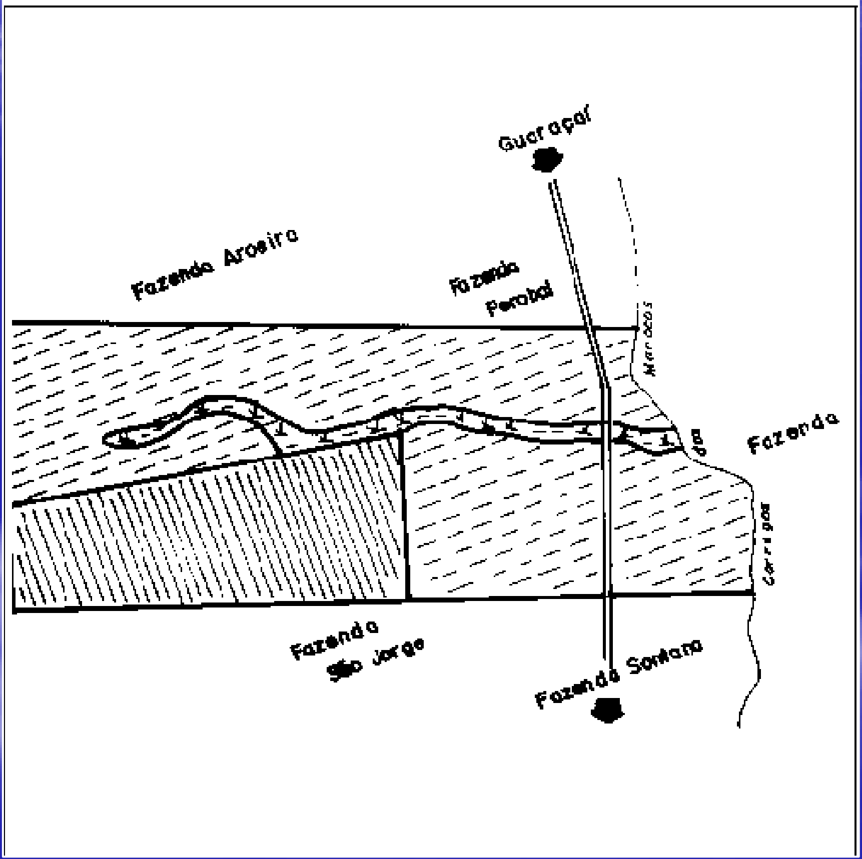
Texture extraction (1)

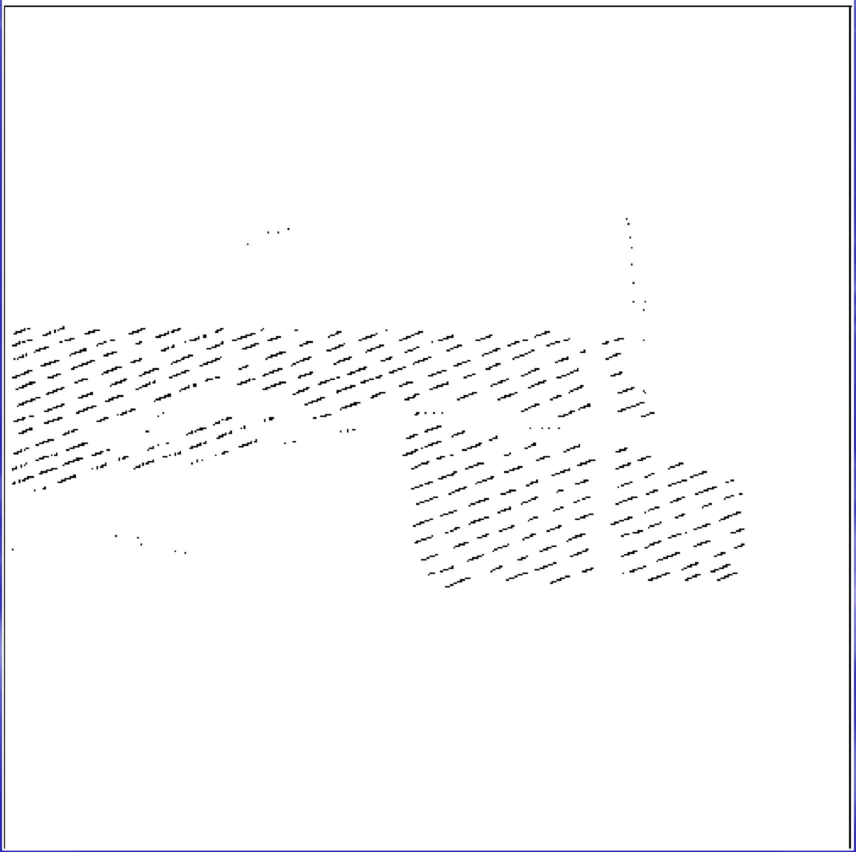




Training images

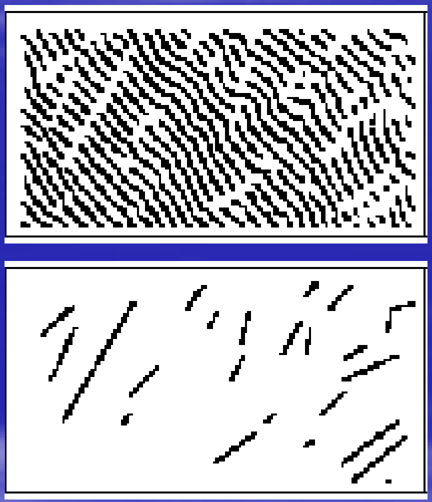
Texture extraction (2)



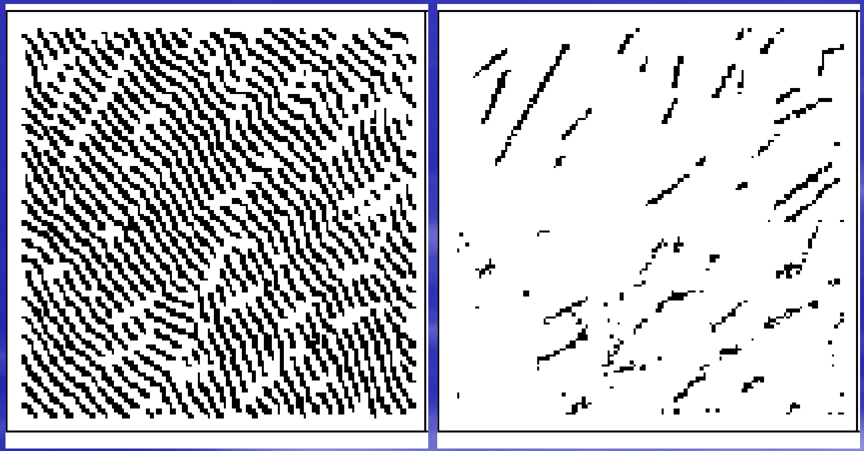


Test images

Fracture Detection

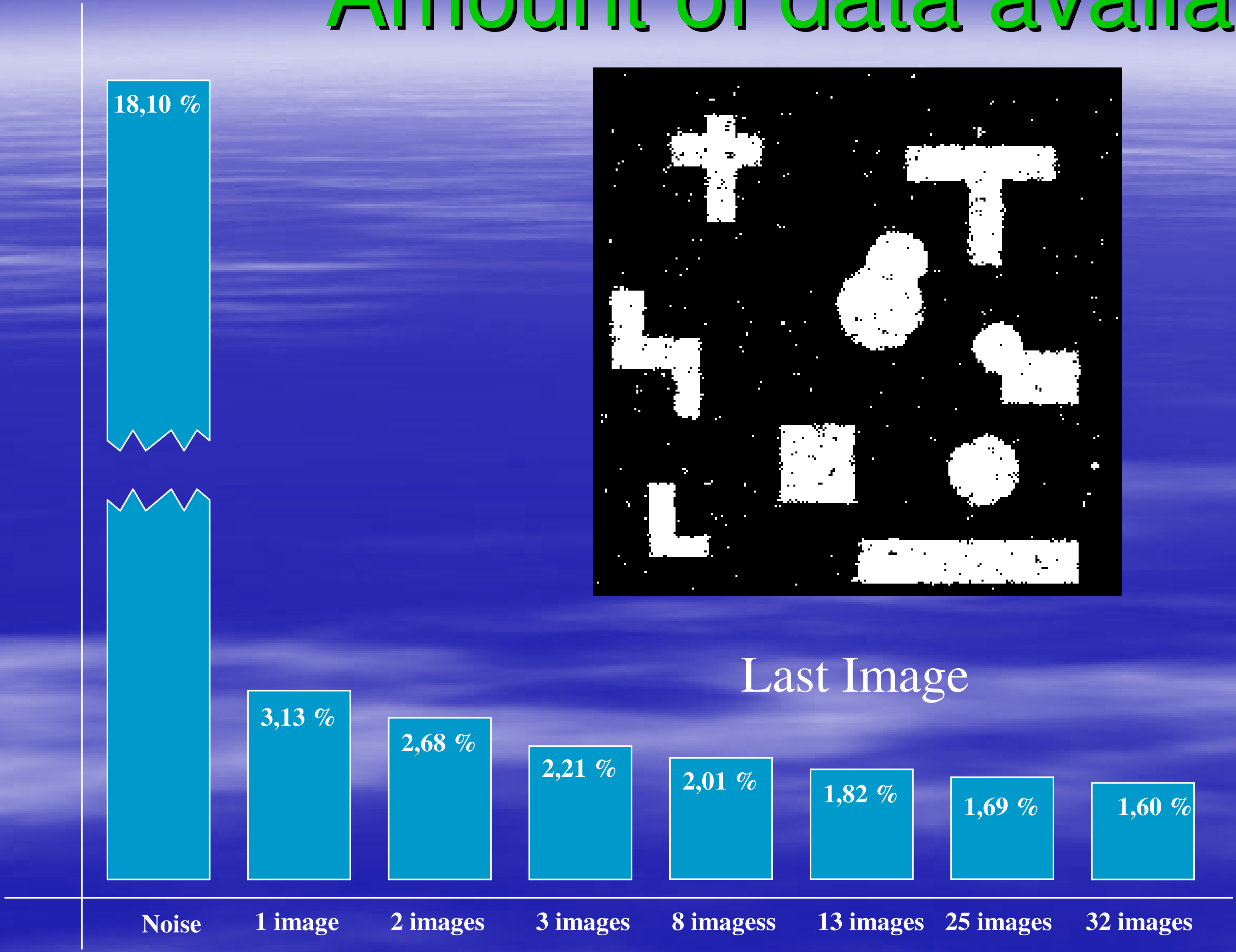


Training images



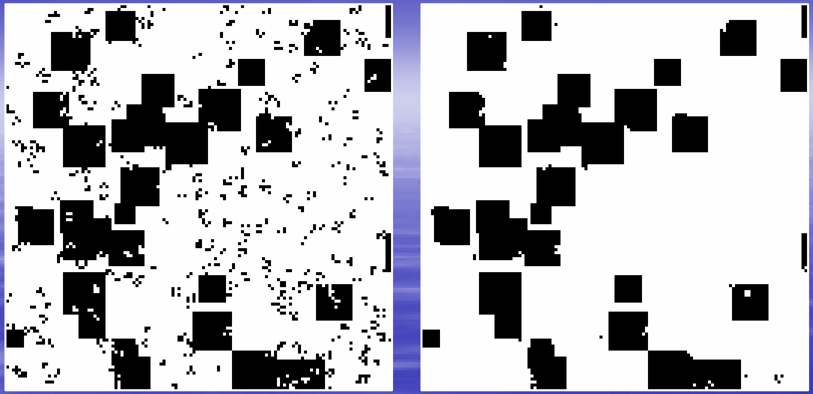
Test images

Amount of data available

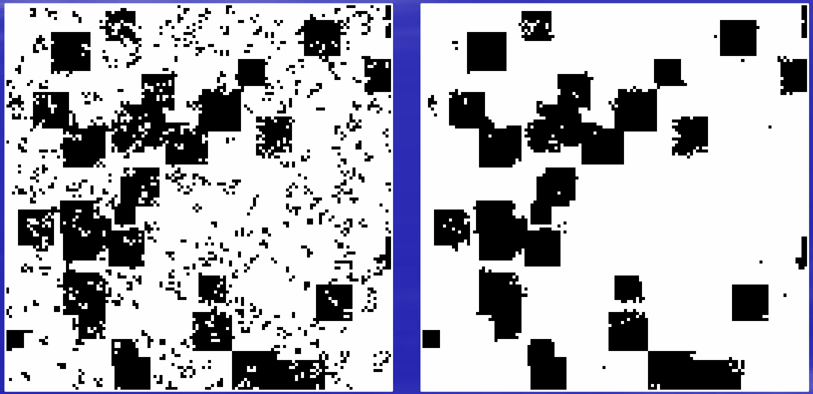


Last Image

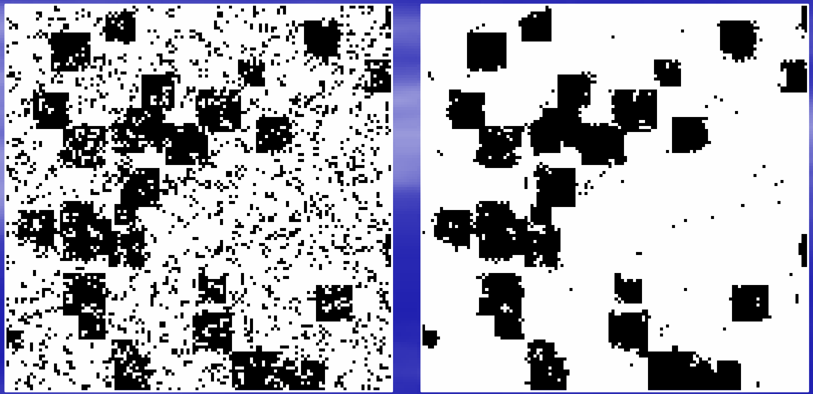
window 5x5, 6 training images



addition: 2%
subtraction: 1%
distict patterns : 140.060
in 1.548.384

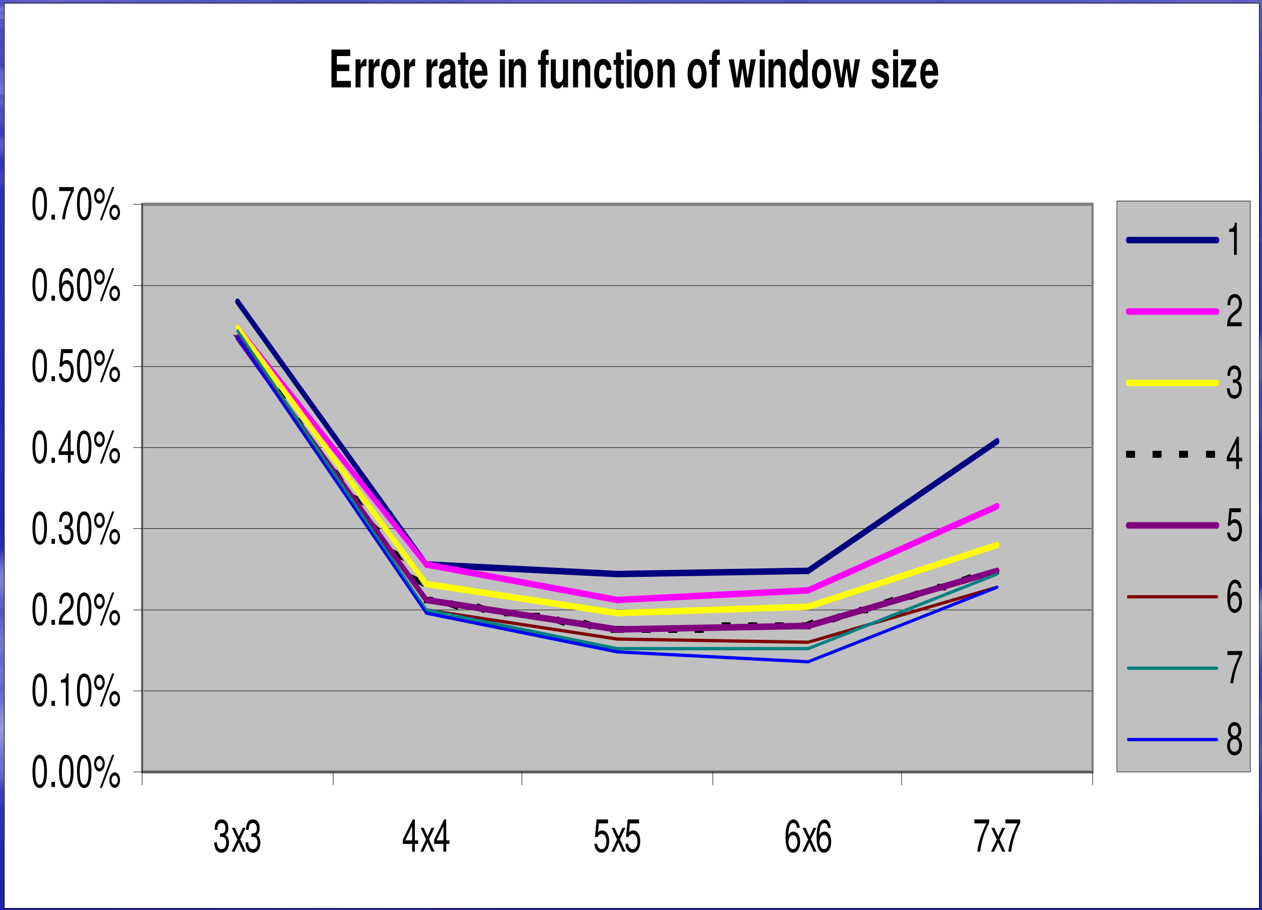


addition: 3%
subtraction: 3%
distinct patterns : 266.743
em 1.548.384



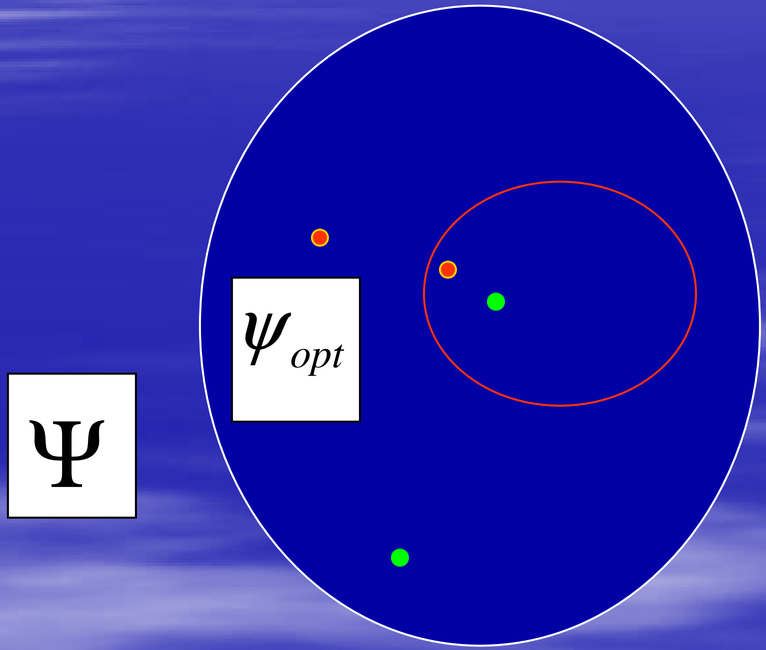
addition: 6%
subtraction: 6%
distinct patterns: 487.494
in 1.548.384

Size of the window

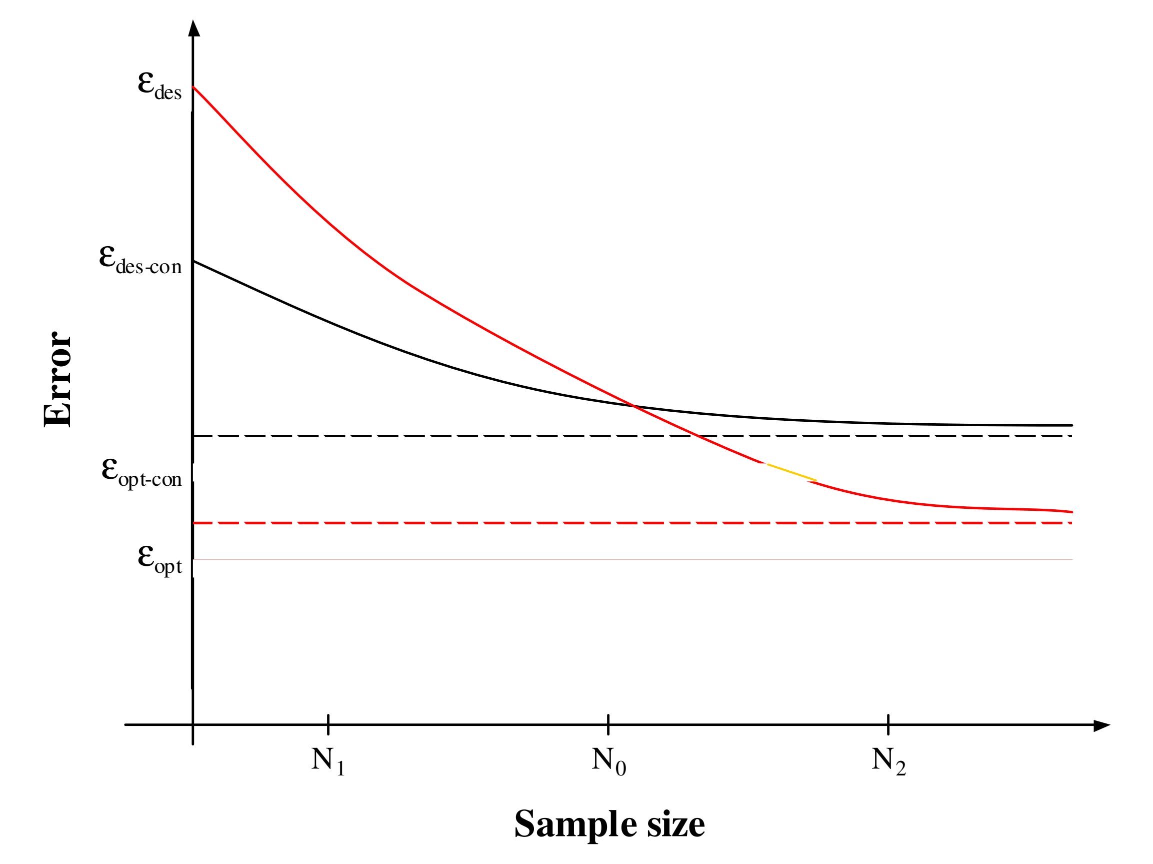


Constraint Design

Constraints



Constraints



Constraints

- **Structural Constraints**
  - impose maximum number of elements in the basis
  - use alternative structural representations (e.g., sequential)

Constraints

- **Algebraic constraints**

  - consider a class of operators satisfying a given algebraic property (e.g., increasingness , idempotence, auto-dualism, etc)

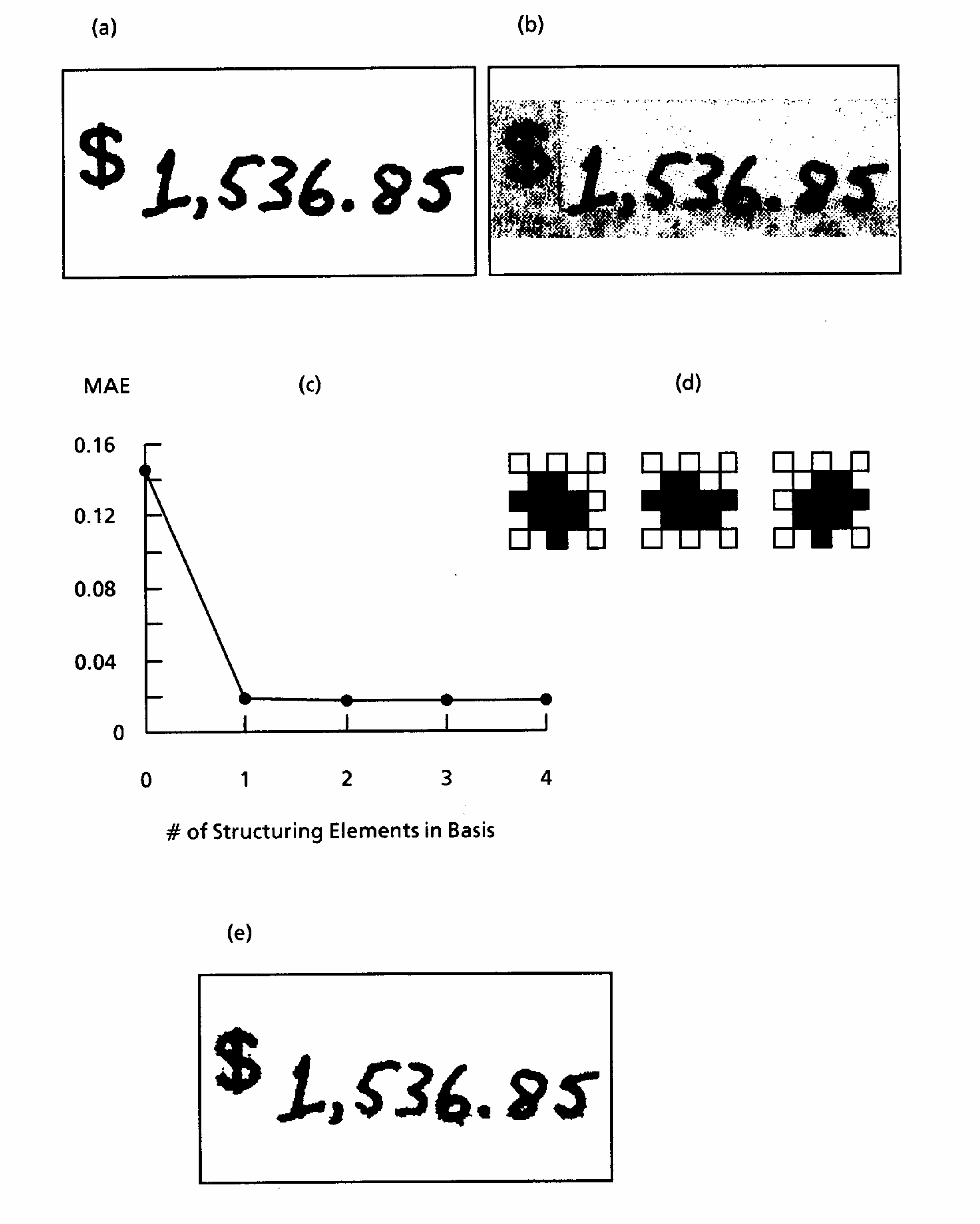
  - consider a constraint by a multi-resolution criteria

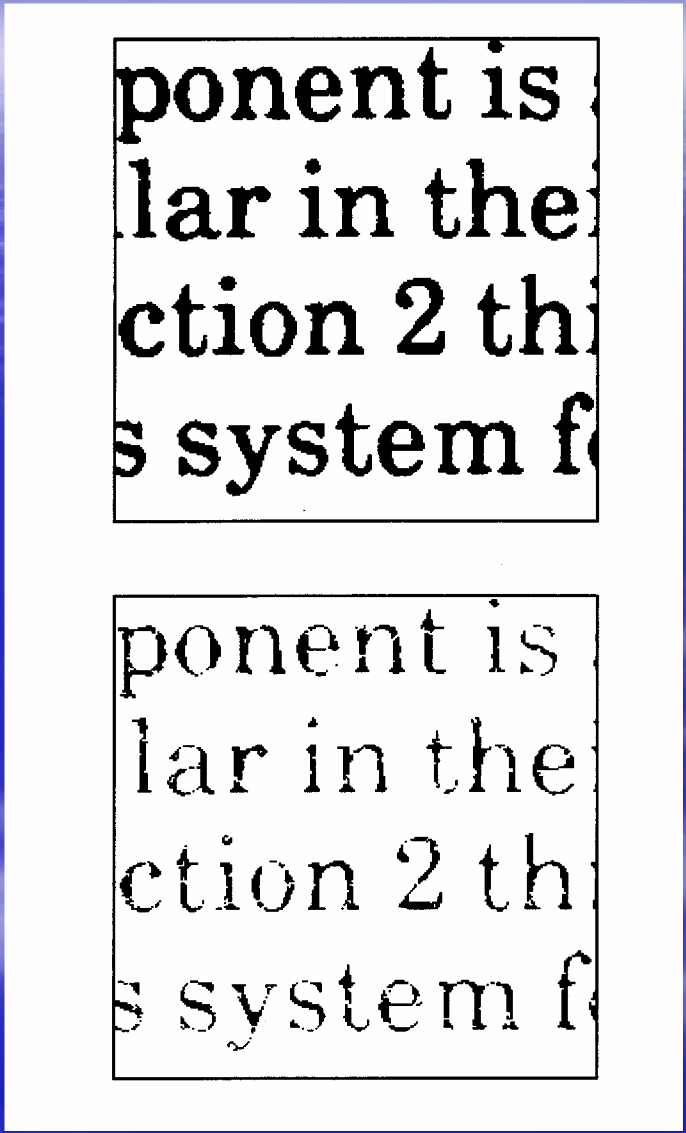
  - consider a constraint by an envelope of operators

  - consider a constraint by a multi-resolution envelope criteria

Structural Constraint

Minimum number of intervals in the basis


(a)
(b)
$ 1,536.85
MAE
(c)
(d)
0.16
0.12
0.08
0.04
0
0
1
2
3
4
# of Structuring Elements in Basis
(e)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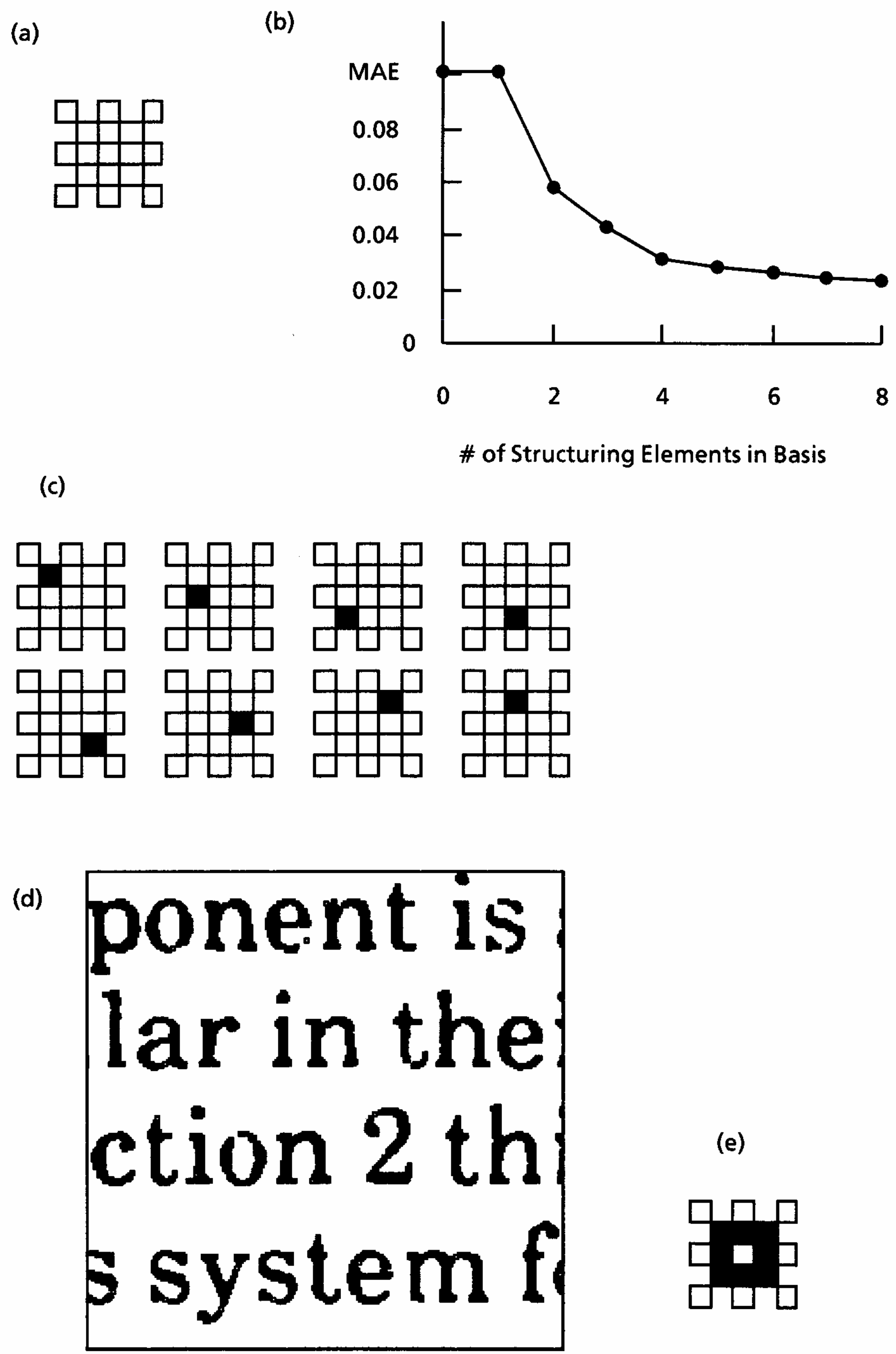

(a)
(b)
MAE
0.08
0.06
0.04
0.02
0
0
2
4
6
8
# of Structuring Elements in Basis
(c)
(d)
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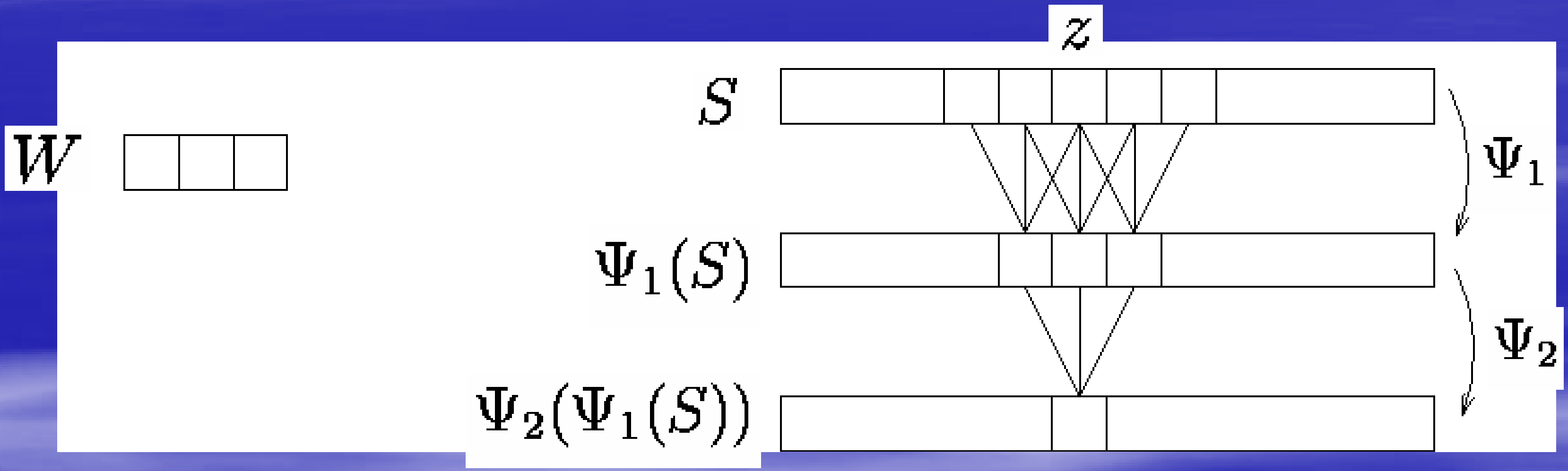
(e)

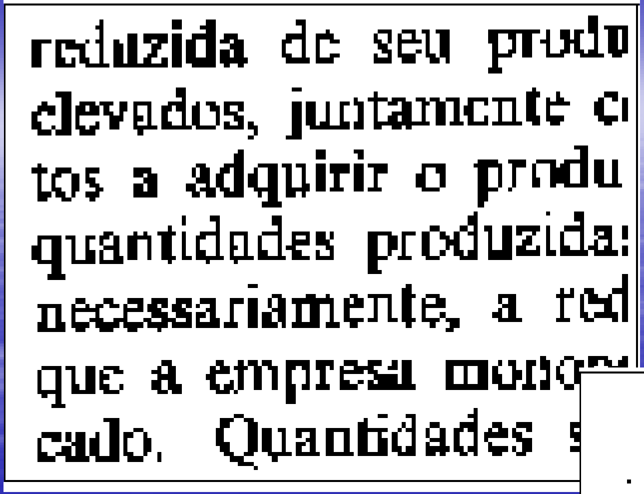
Iterative design

- ***Motivation*** **: composition of operators over small windows produces an operator over a larger window**



- $\Psi = \Psi_2(\Psi_1)$ is a $W \oplus W$-operator

Application example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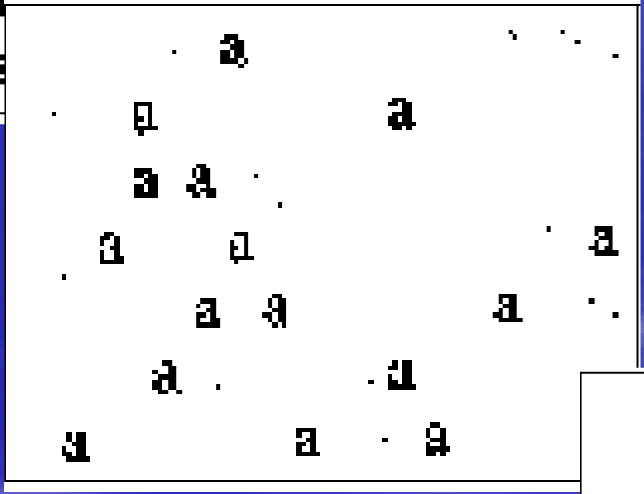
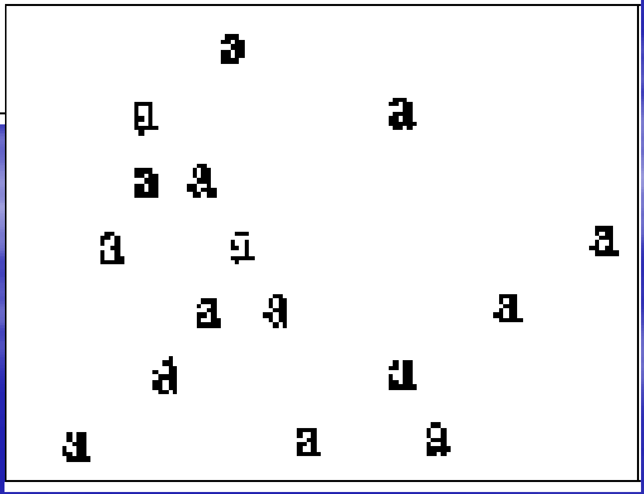
test image

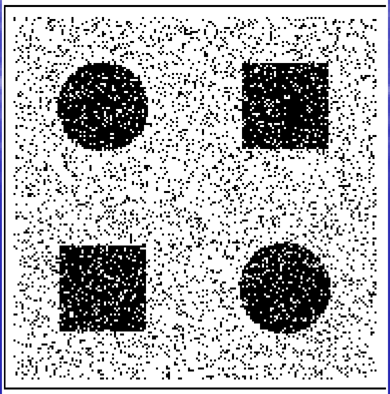


iteration 1

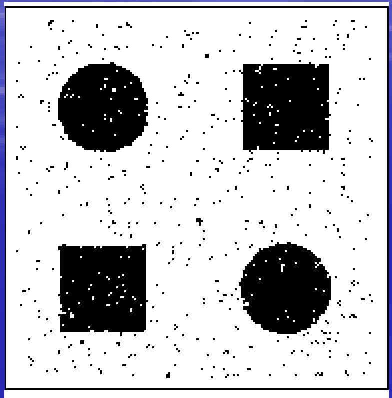


iteration 2

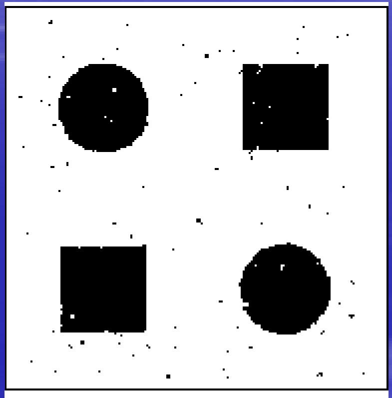
Application example



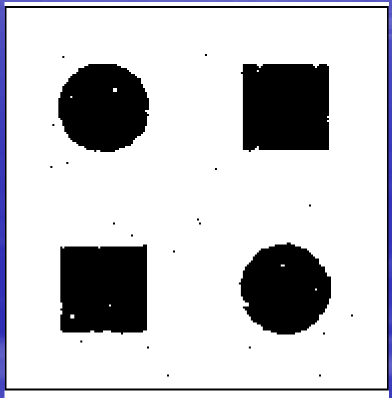
test image



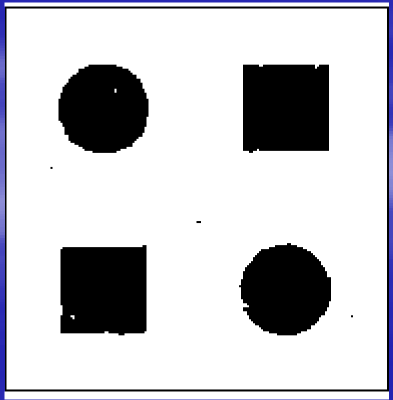
iteration 1



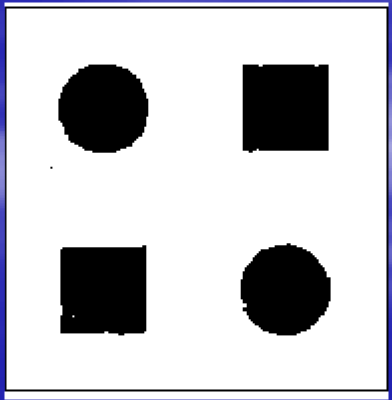
iteration 2



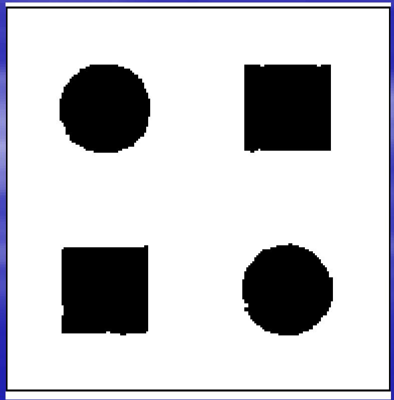
iteration 3



iteration 4



iteration 5

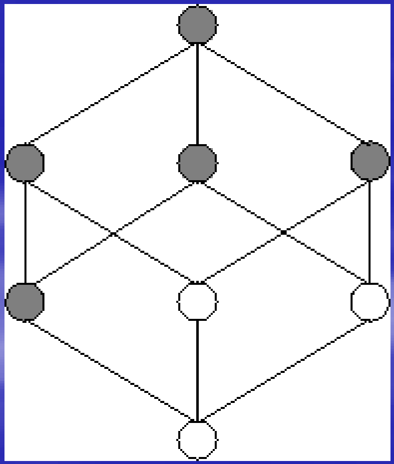


iteration 6

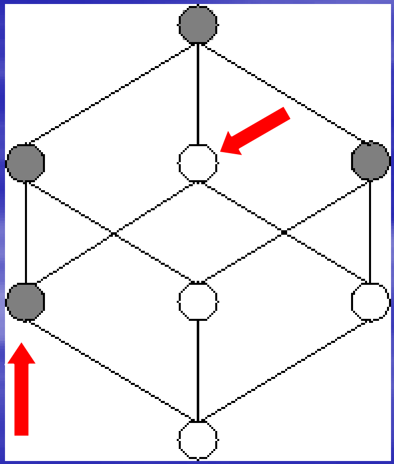
Algebraic Constraint

Increasing *W*-operators

$$x \leq y \Rightarrow \psi(x) \leq \psi(y)$$

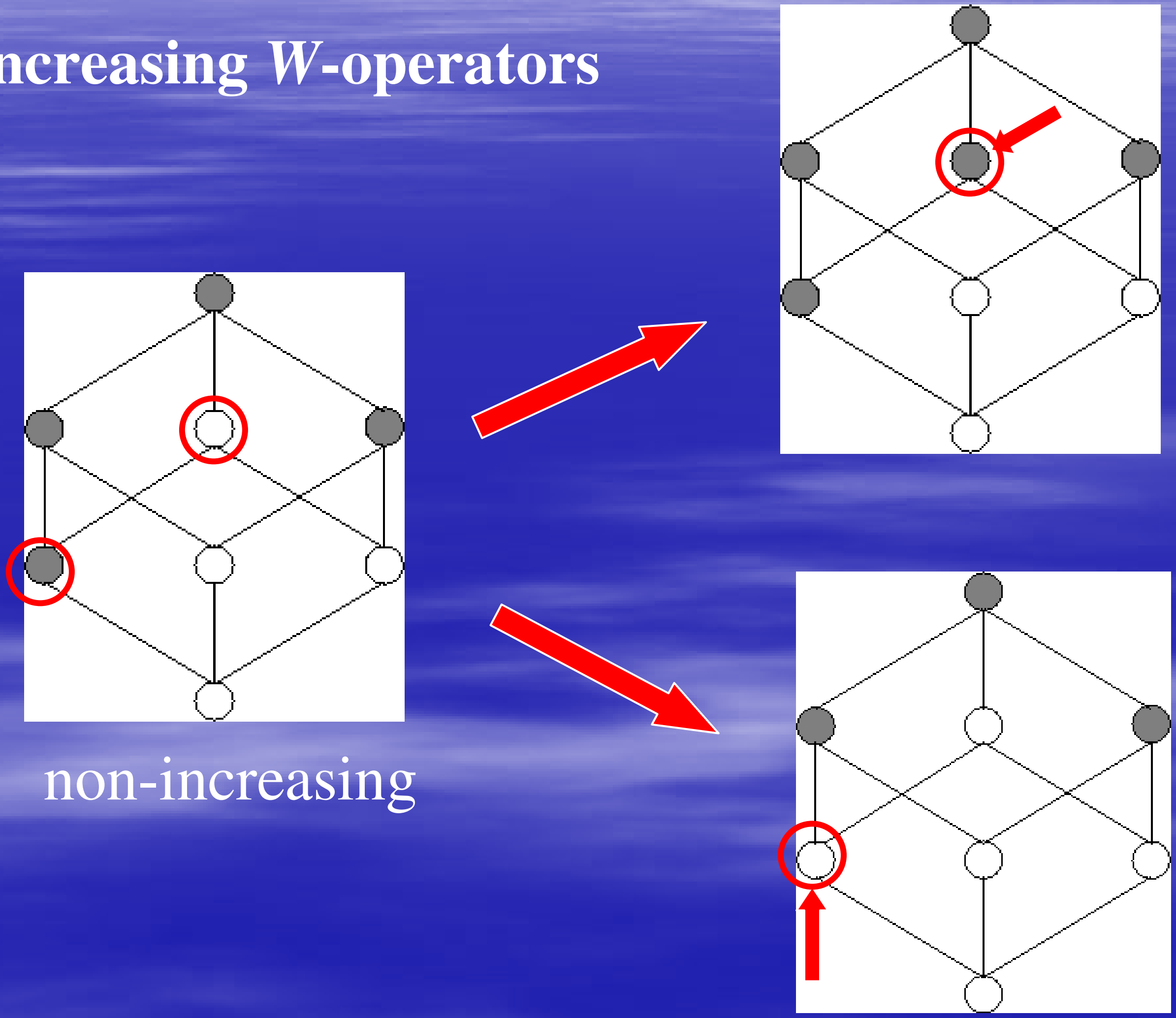


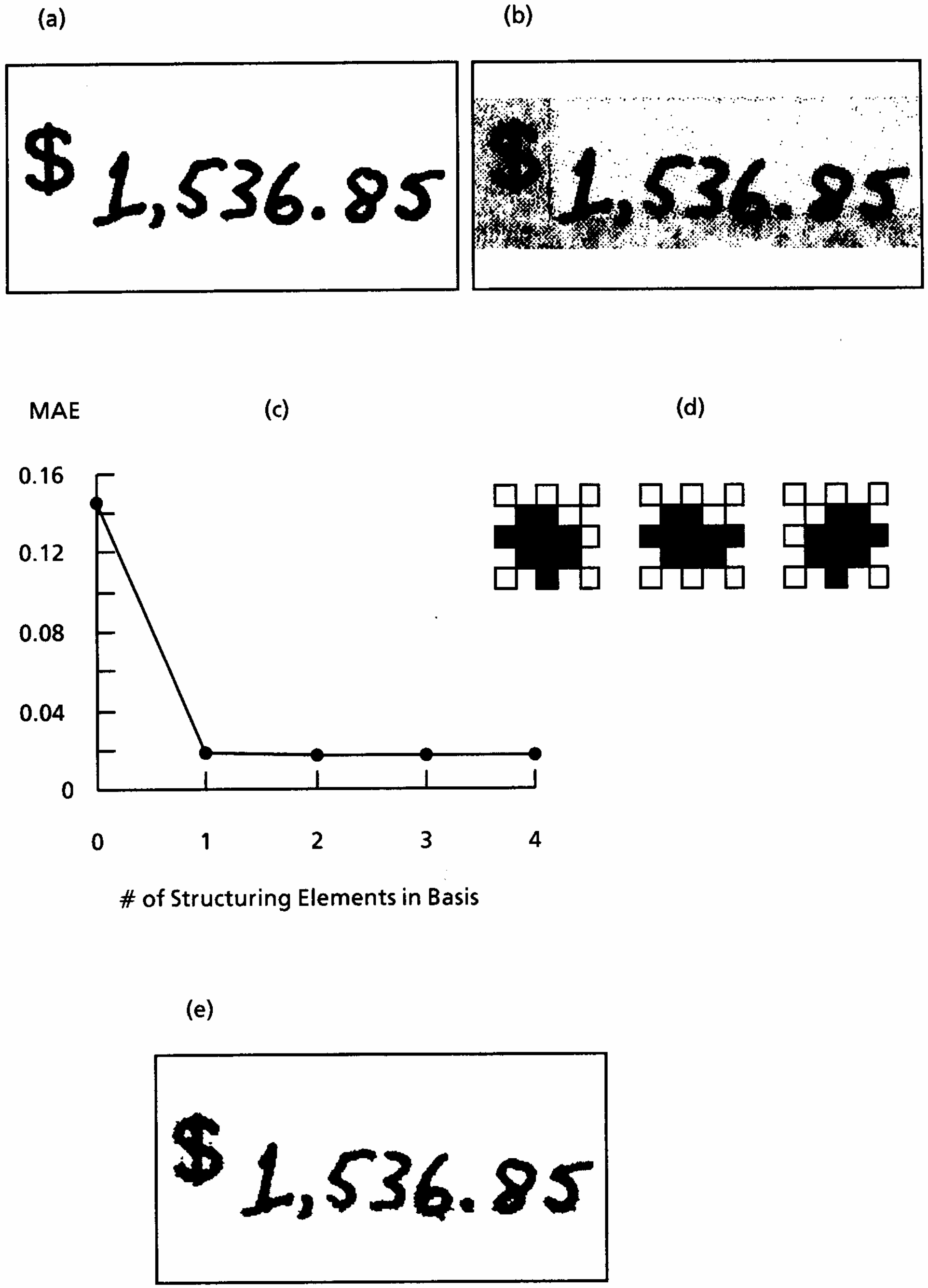
increasing



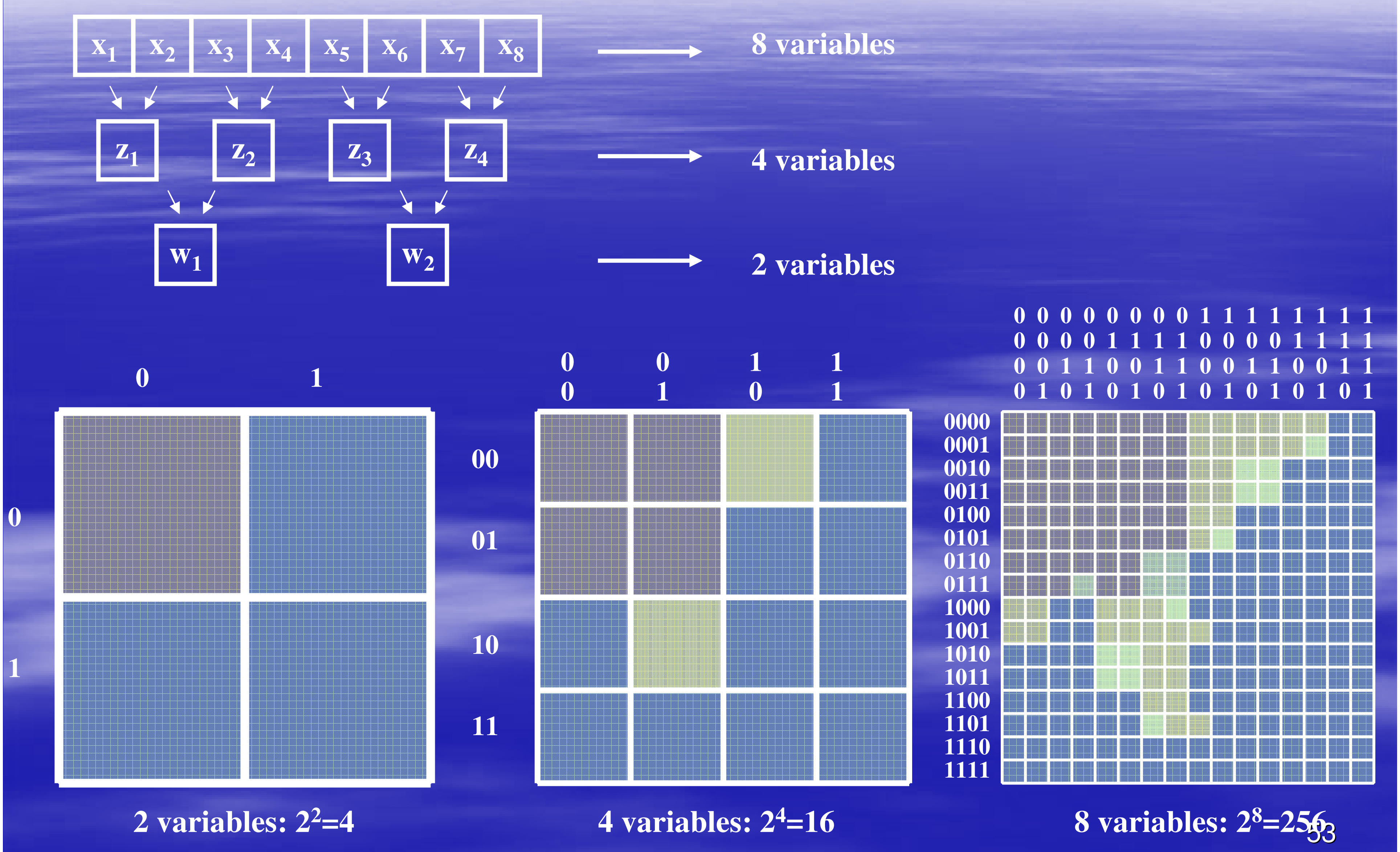
non-increasing

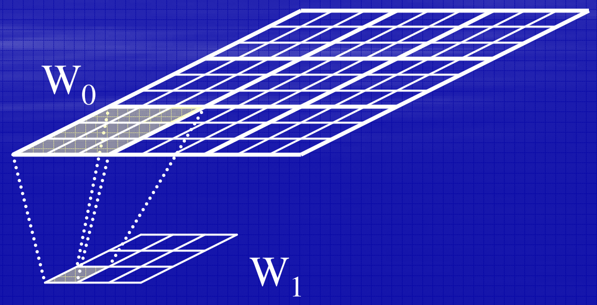
Design of increasing *W*-operators




(a)
(b)
MAE
(c)
(d)
0.16
0.12
0.08
0.04
0
0
1
2
3
4
# of Structuring Elements in Basis
(e)

Multi-resolution constraint


x1 x2 x3 x4 x5 x6 x7 x8
8 variables
z1 z2 z3 z4
4 variables
w1 w2
2 variables
0 1
0
1
2 variables: 2^2=4
0 0 1 1
0 1 0 1
00
01
10
11
4 variables: 2^4=16
0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1
0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1
0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1
0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111
8 variables: 2^8=256



$D_1 = P(W_1)$

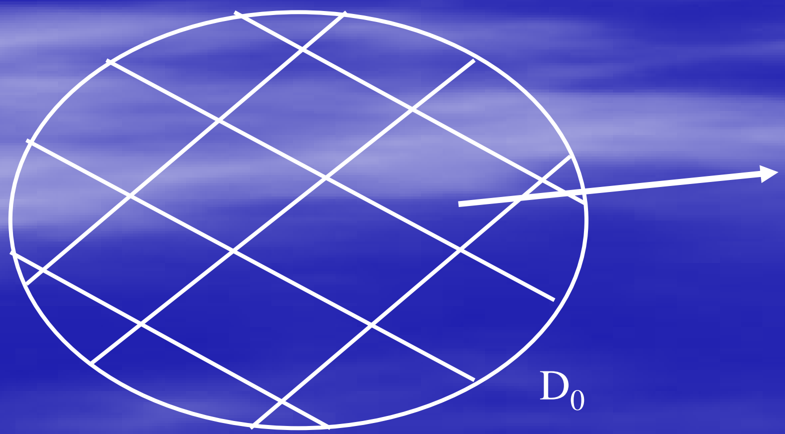
$D_0 = P(W_0)$

$\mathbf{z}_i = p_i(\mathbf{x}_{i1}, \ldots, \mathbf{x}_{i9})$, $\mathbf{z} = p(\mathbf{x})$, $p=(p_1, .. p_9)$

Let $\phi: D_1 \rightarrow \{0,1\}$, it defines the operator Ψ_ϕ on D_o by

$\Psi_\phi (\mathbf{x}) = \phi(p(\mathbf{x}))$

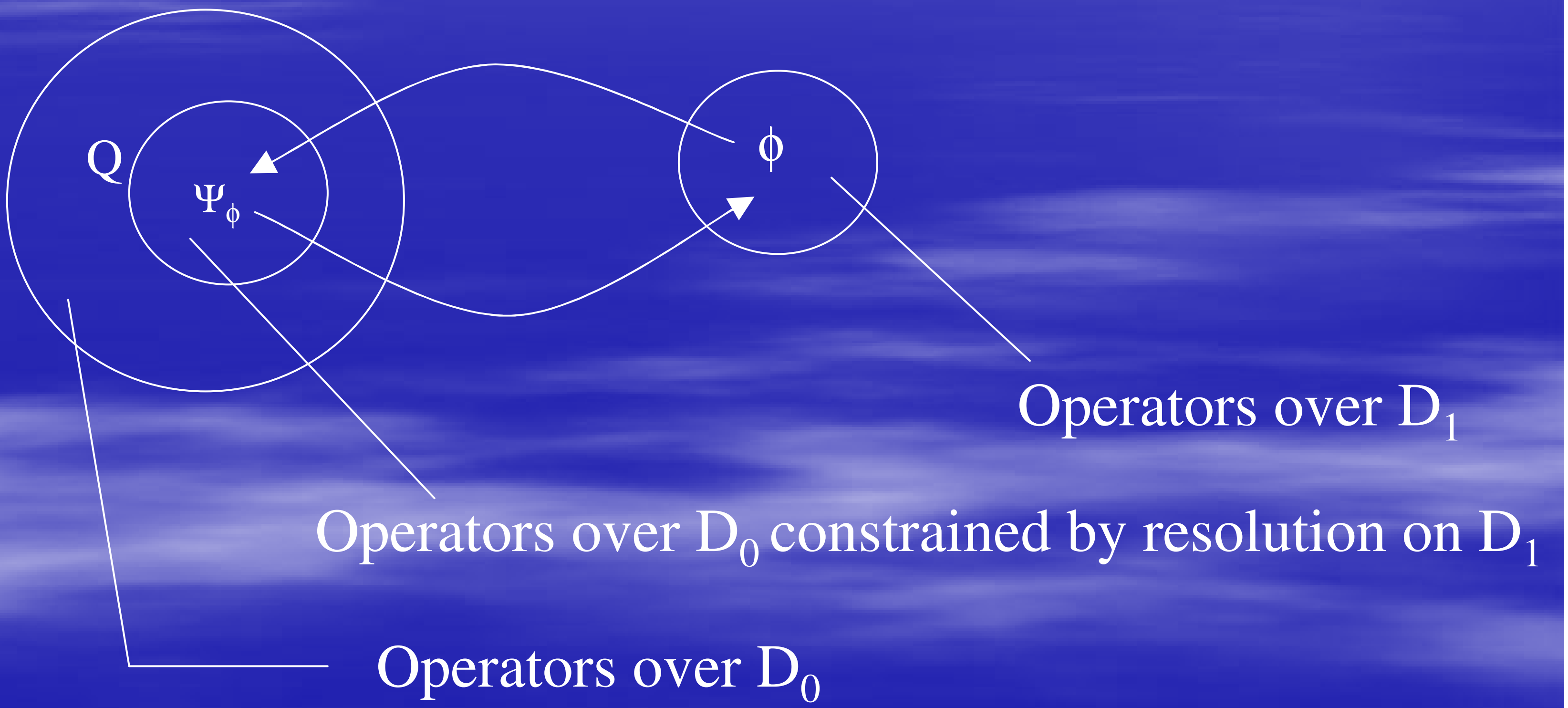
The operador Ψ_ϕ is constrained by resolution to D_1



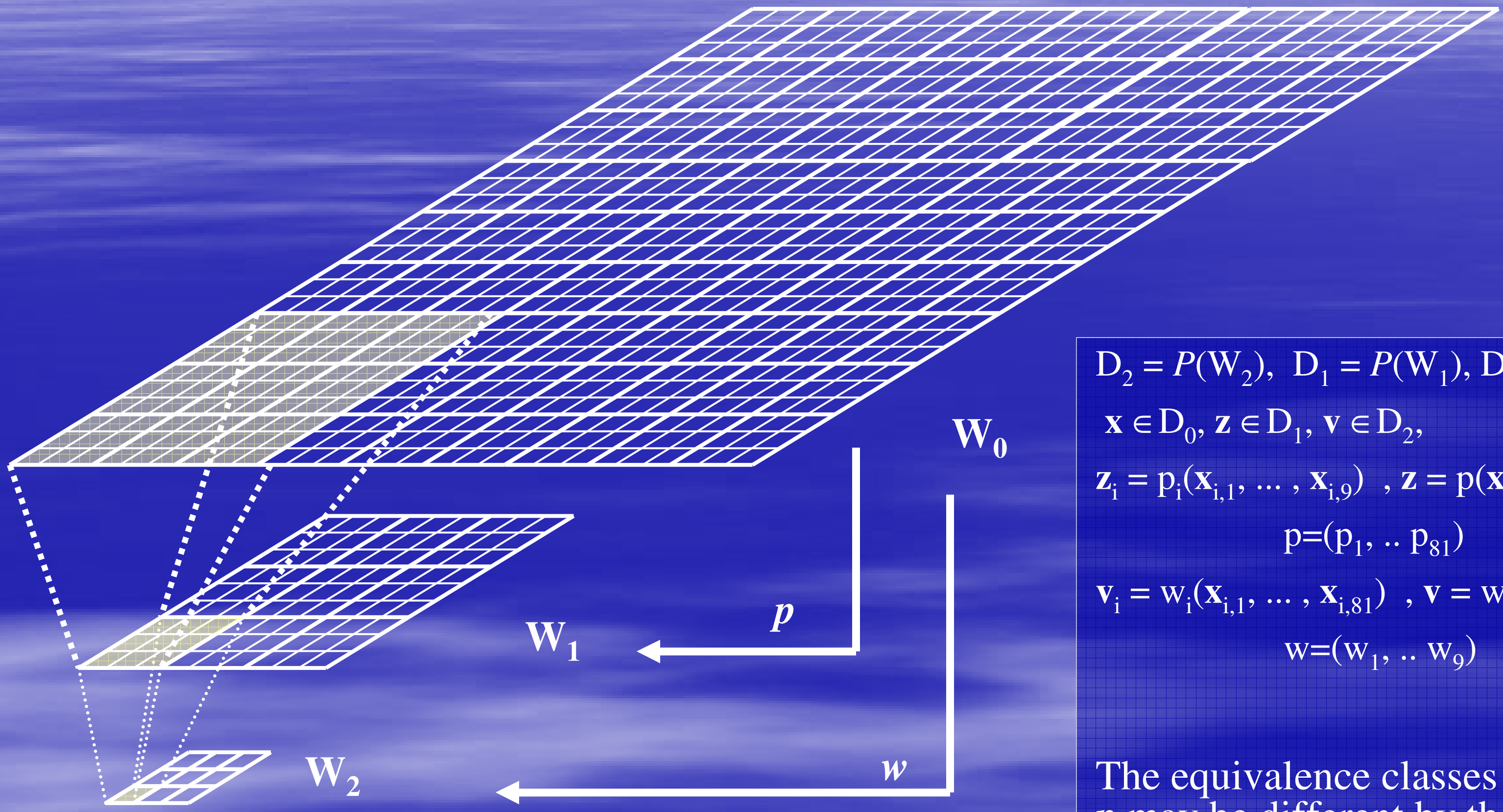
Equivalence classes defined by

$p(\mathbf{x}) = p(\mathbf{y})$

Properties



Q
Ψ_ϕ
ϕ
Operators over D_1
Operators over D_0 constrained by resolution on D_1
Operators over D_0



$D_2 = P(W_2)$, $D_1 = P(W_1)$, $D_0 = P(W_0)$

$\mathbf{x} \in D_0$, $\mathbf{z} \in D_1$, $\mathbf{v} \in D_2$,

$\mathbf{z}_i = p_i(\mathbf{x}_{i,1}, \ldots , \mathbf{x}_{i,9})$, $\mathbf{z} = p(\mathbf{x})$,

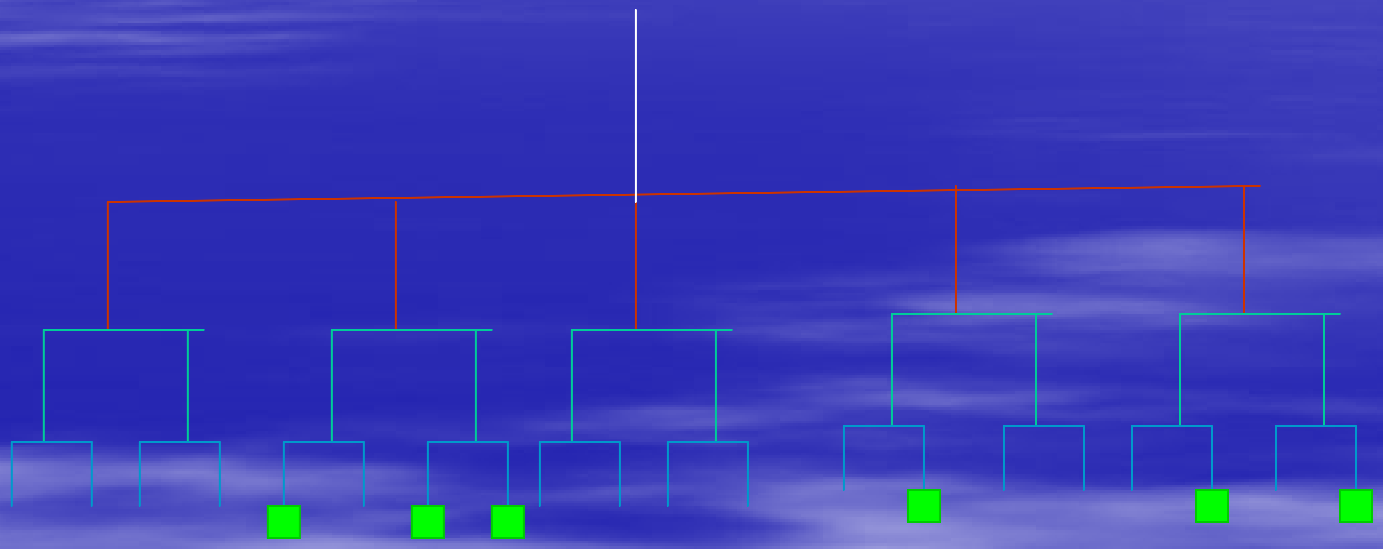
$p=(p_1, .. \, p_{81})$

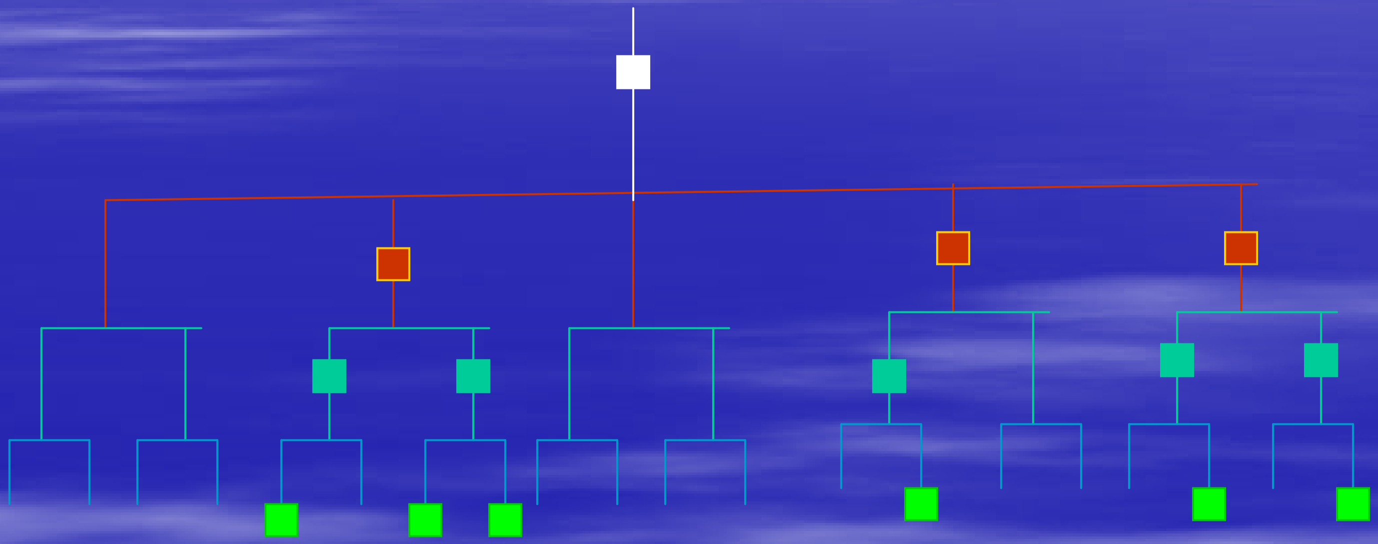
$\mathbf{v}_i = w_i(\mathbf{x}_{i,1}, \ldots , \mathbf{x}_{i,81})$, $\mathbf{v} = w(\mathbf{x})$,

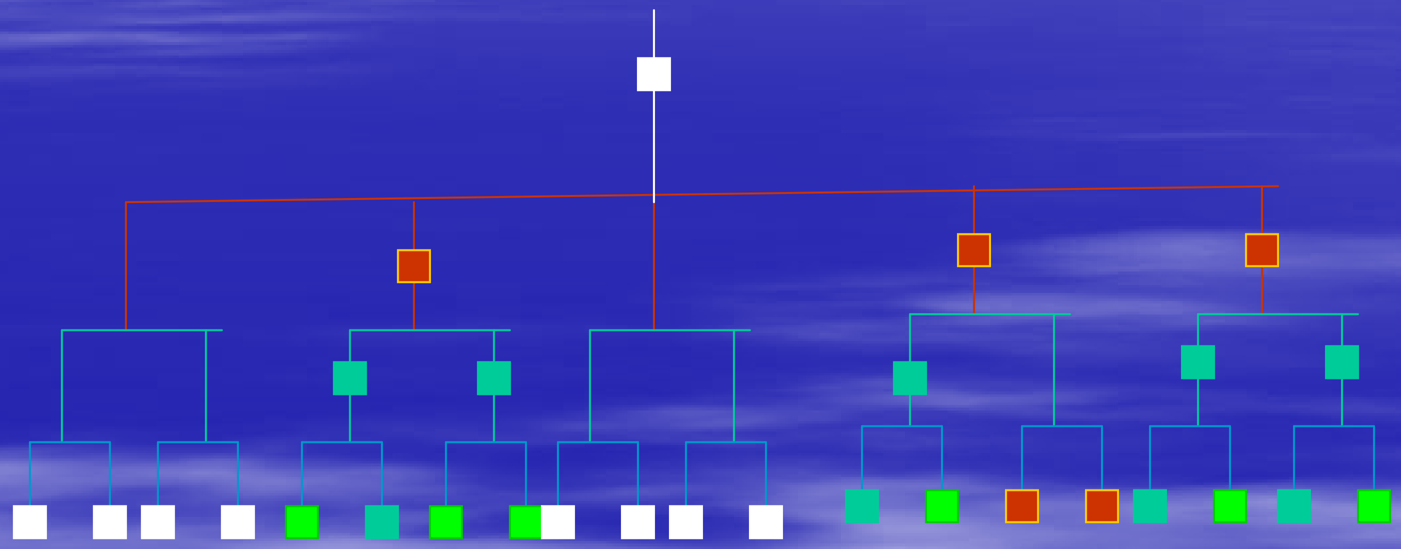
$w=(w_1, .. \, w_9)$

The equivalence classes defined by ***p*** may be different by the ones defined by ***w***.

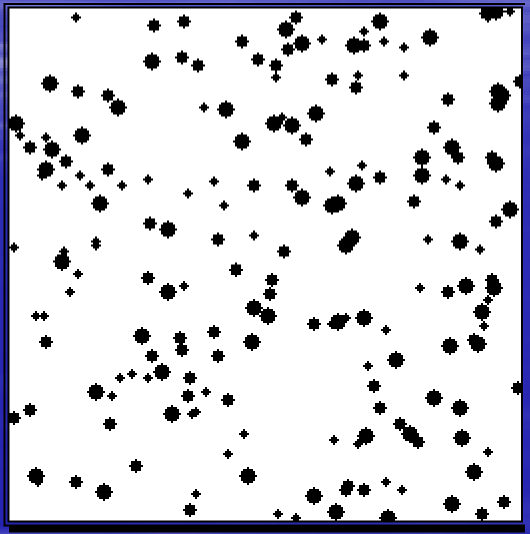
$$\psi(\mathbf{x}) = \begin{cases} \psi_{0,N}(\mathbf{x}) & \mathit{if} \quad N(\mathbf{x}) > 0 \\ \psi_{1,N}(\mathbf{x}) & \mathit{if} \quad N(\mathbf{x}) = 0, N(\rho_1(\mathbf{x})) > 0 \\ \vdots & \\ \psi_{m-1,N}(\mathbf{x}) & \mathit{if} \quad N(\mathbf{x}) = 0, \ldots, N(\rho_{m-2}(\mathbf{x})) = 0, N(\rho_{m-1}(\mathbf{x})) > 0 \\ \psi_{m,N}(\mathbf{x}) & \mathit{if} \quad N(\mathbf{x}) = 0, \ldots, N(\rho_{m-1}(\mathbf{x})) = 0, N(\rho_m(\mathbf{x})) > 0 \end{cases}$$



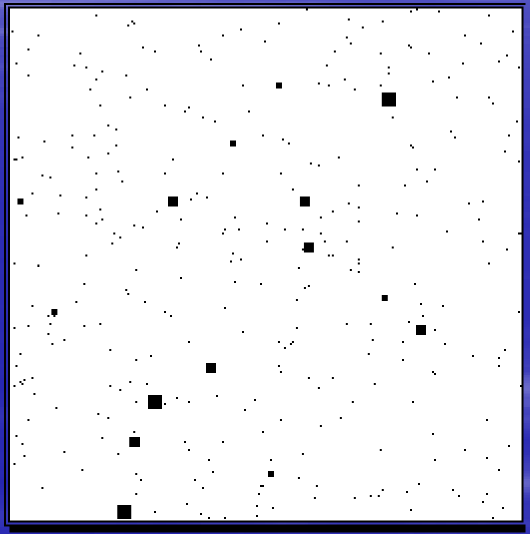




Multiresolution Noise



image



noise

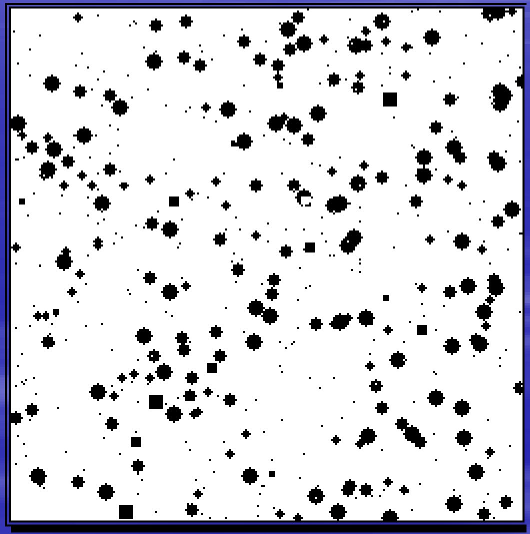
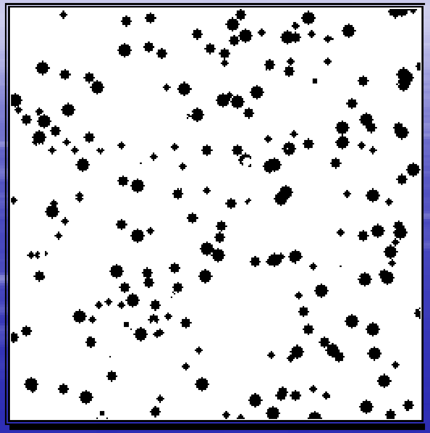
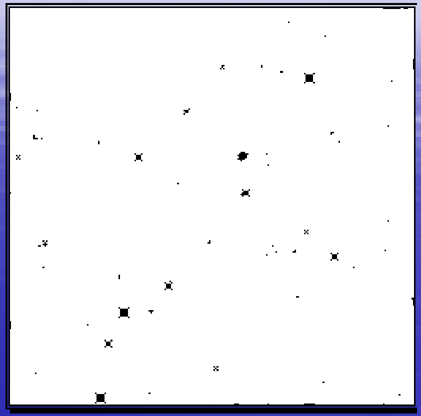


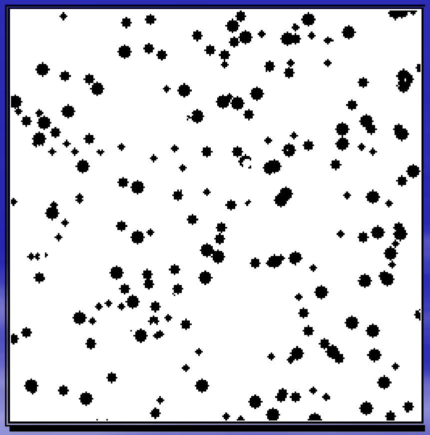
image + noise

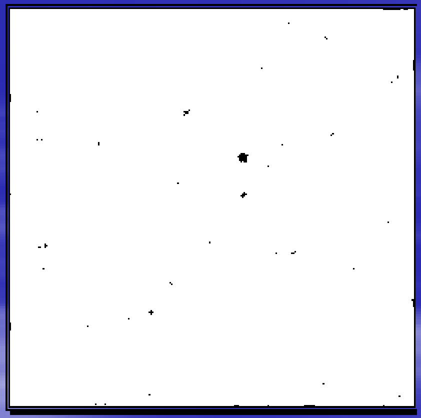
3x3 window





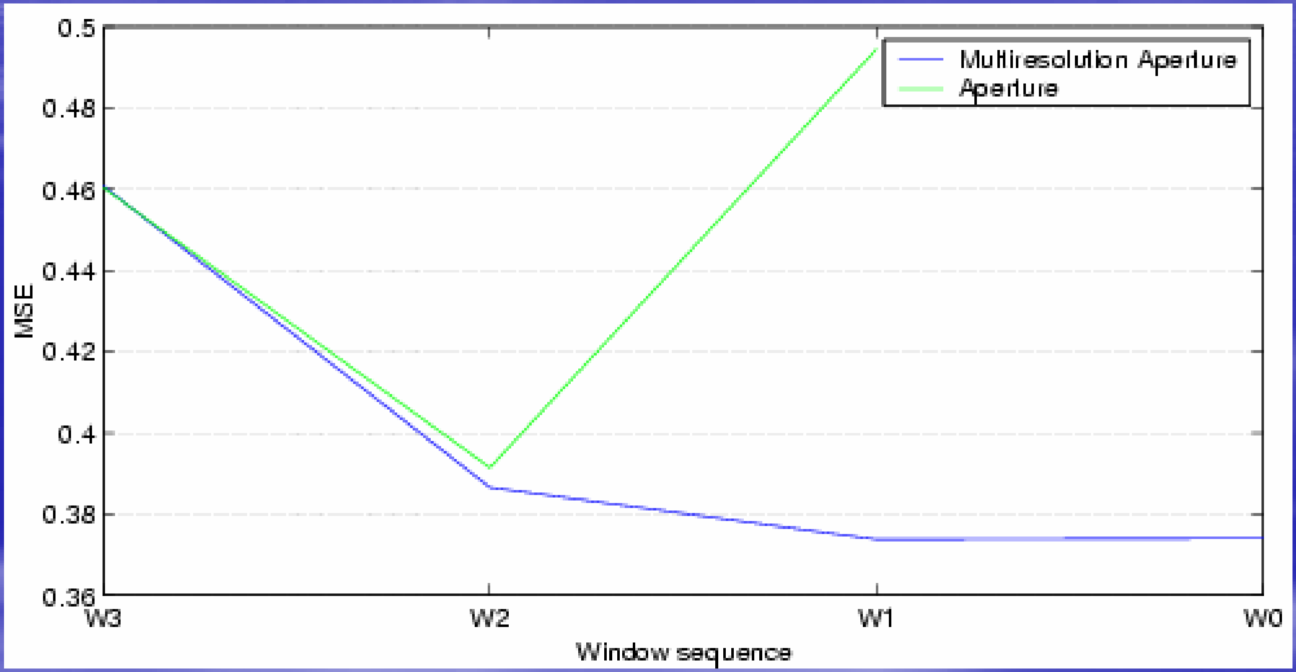
Pyramid



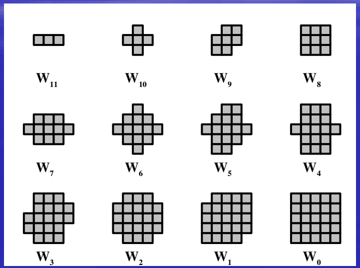


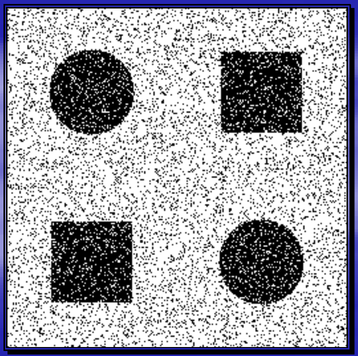
Restauration

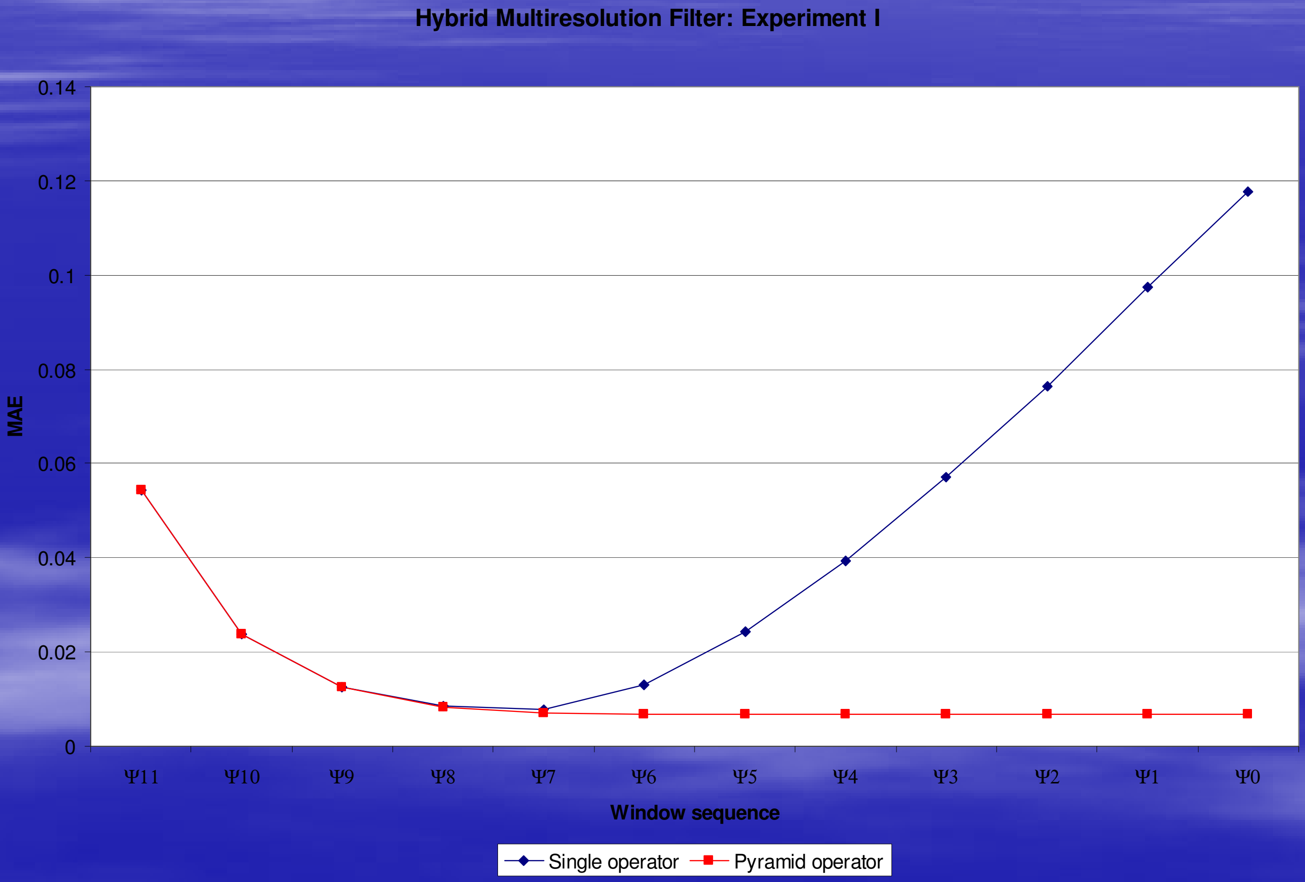
Persisting noise


0.5
0.48
0.46
0.44
0.42
0.4
0.38
0.36
MSE
W3
W2
W1
W0
Window sequence
Multiresolution Aperture
Aperture

Example


W11 W10 W9 W8
W7 W6 W5 W4
W3 W2 W1 W0




Hybrid Multiresolution Filter: Experiment I
MAE
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
Ψ11 Ψ10 Ψ9 Ψ8 Ψ7 Ψ6 Ψ5 Ψ4 Ψ3 Ψ2 Ψ1 Ψ0
Window sequence
Single operator
Pyramid operator

Envelope constraint

Independent Constraints

Constraints

Restriction of

the operators space

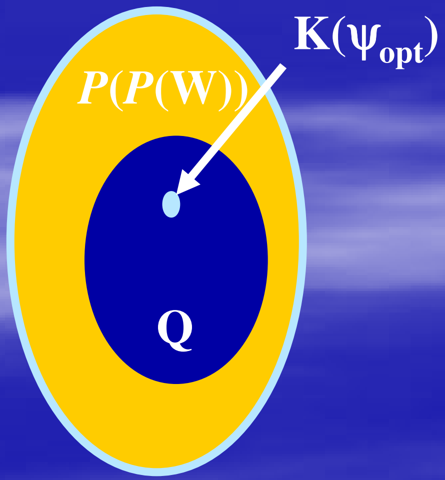
$K(\psi_{opt}) \in Q \subseteq P(P(W))$

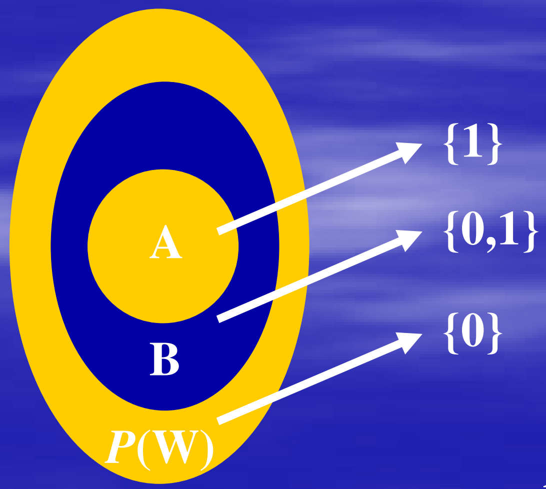
Independent Constraint

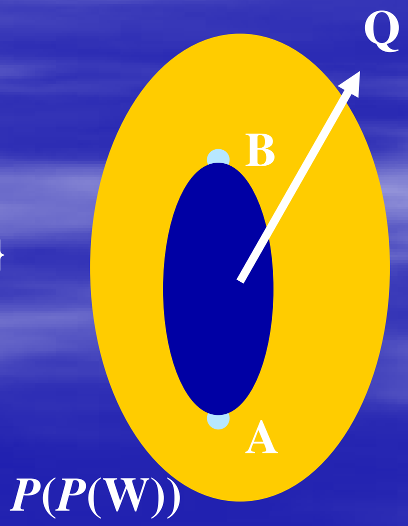
Let be $A,B \subseteq P(W)$ with $A \subseteq B$:

$h_\psi(x)=1 \ \forall\ x \in A$ & $h_\psi(x)=0 \ \forall x \notin B$,

$\forall \psi : K(\psi) \in Q$







Independent Constraints

<u>Proposition</u>: if Q is an independent restriction then exist a par of operators

(α,β) such that, for any $\psi \in \Psi_W$

$$K(\psi) \in Q \Leftrightarrow \alpha \leq \psi \leq \beta$$

where $K(\alpha) = A$ and $K(\beta) = B$

- All independent constraint is characterized by two operators α and β
- The pair (α,β) is called "Envelope"

Definition

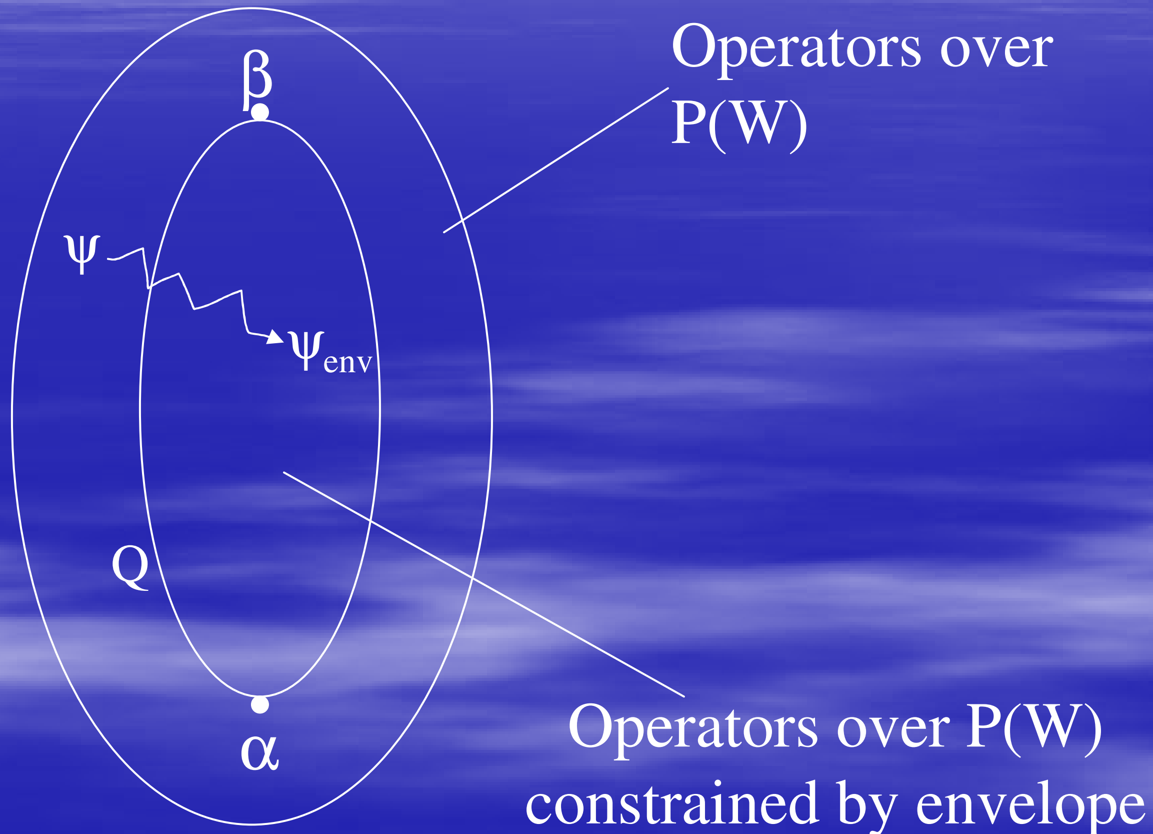
Desing of two heristic filters α and β such that when we know that $\alpha \leq \psi_{opt} \leq \beta$, the restriction is defined by:

$$\mathbf{Q} = \{ \psi : \alpha \leq \psi \leq \beta \}$$

and any filter ψ can be projected into the restriction by

$$\psi_{env} = (\psi \vee \alpha) \wedge \beta$$

Definition



β
ψ
ψenv
Q
α
Operators over P(W)
Operators over P(W) constrained by envelope

Properties

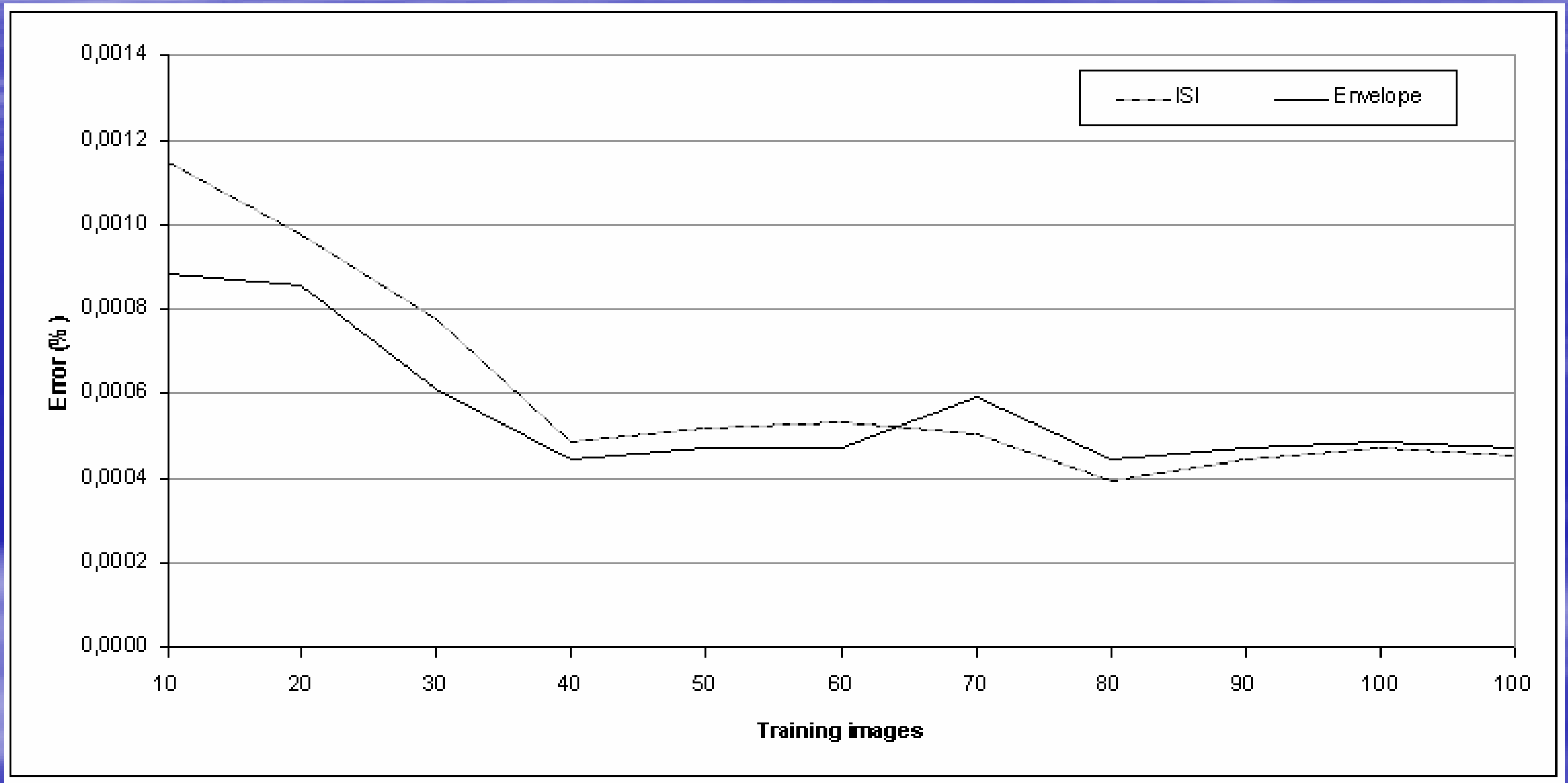
- $(\psi_{opt} \vee \alpha) \wedge \beta$ is optimal in **Q**.

- If $\alpha \leq \psi_{opt} \leq \beta$ then $\mathrm{Error}[\psi_{env}] \leq \mathrm{Error}[\psi]$

- If $\alpha \leq \psi_{opt} \leq \beta$ is not true, then

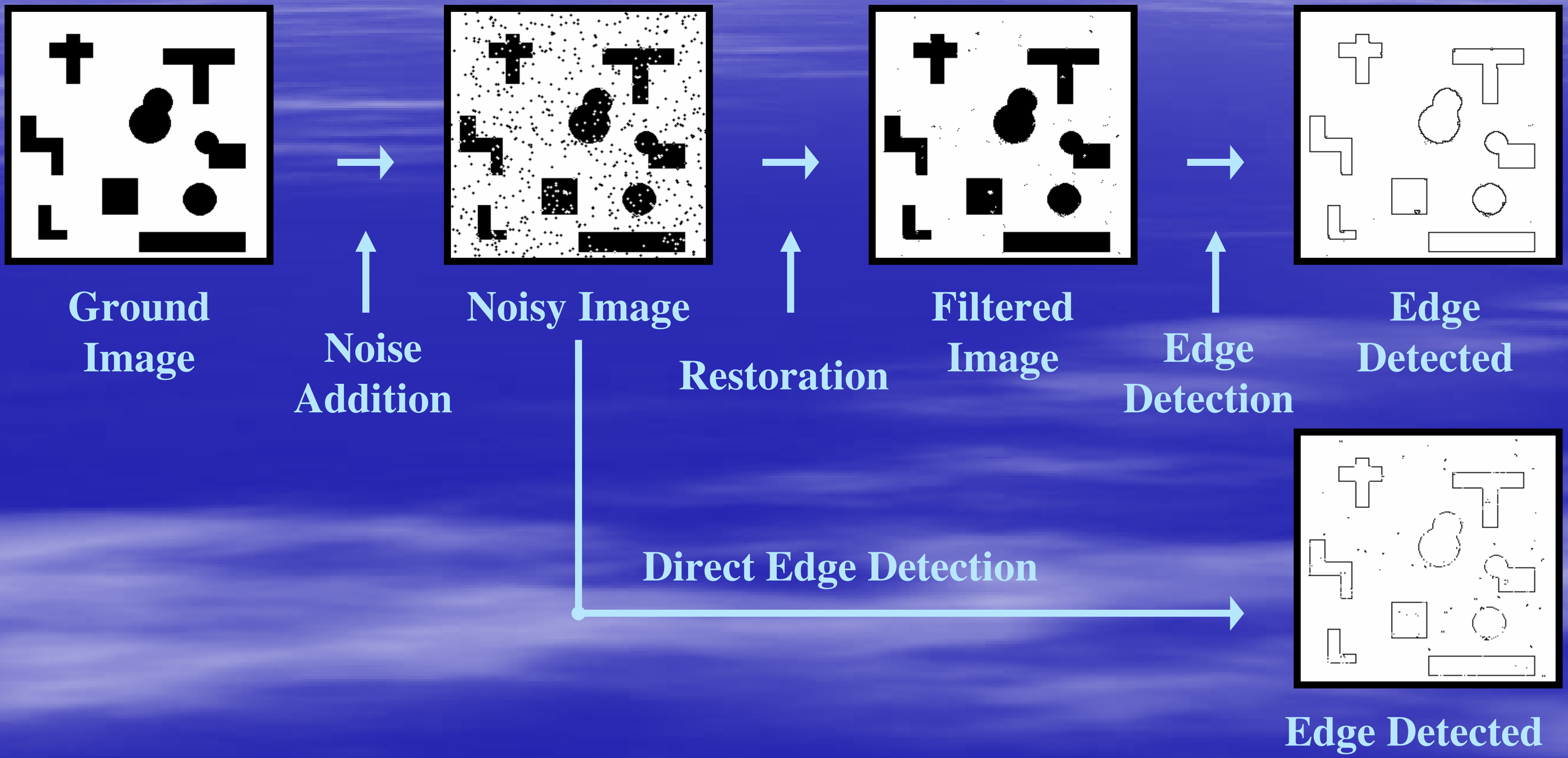
$$\mathrm{Lim}_{N\to\infty} \mathrm{Error}[\psi_{env,N}] > \mathrm{Lim}_{N\to\infty} \mathrm{Error}[\psi_N]$$

Example



0,0014
0,0012
0,0010
0,0008
0,0006
0,0004
0,0002
0,0000
Error (%)
ISI
Envelope
10
20
30
40
50
60
70
80
90
100
100
Training images

Noise Edge Detection



Restoration → a) **Machine design of the restoration**

ψ_{pac} designed by examples

→ b) **Human-machine design of the restoration**

$$\psi_{con} = (\psi_{pac} \cap \beta) \cup \alpha$$

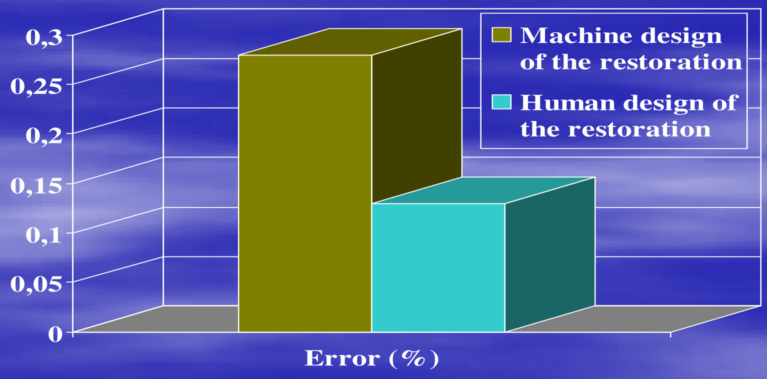
$\alpha = \delta_{B\oplus B}\varepsilon_{B\oplus B}\delta_{B}\varepsilon_{B}$ and $\beta = \varepsilon_{B\oplus B}\delta_{B\oplus B}\varepsilon_{B}\delta_{B}$

α and β are alternating sequential filters with

$$P[\ \alpha(S) \leq I \leq \beta(S)\ ] \approx 1$$

B is the 3x3 square

Machine design of the restoration	Human-Machine design of the restoration
0.28 %	0.13 %



Noise Edge Detection

Edge Detection

→ a) **Machine design over noisy images**

ζ_{pac} designed by examples from noisy images

→ b) **Human design after restoration**

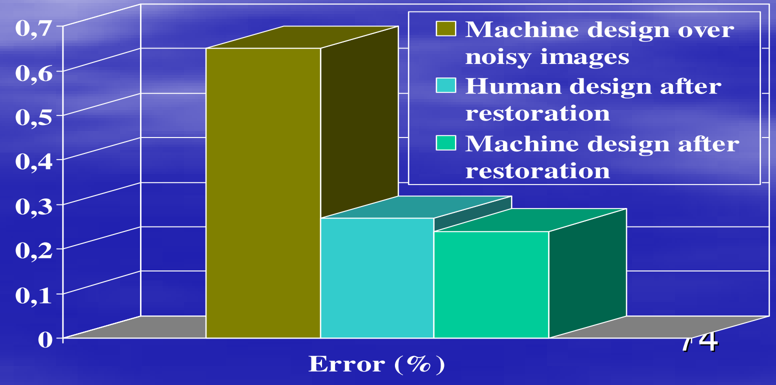
$\zeta = I_d - \varepsilon_B$

B is the 3x3 square

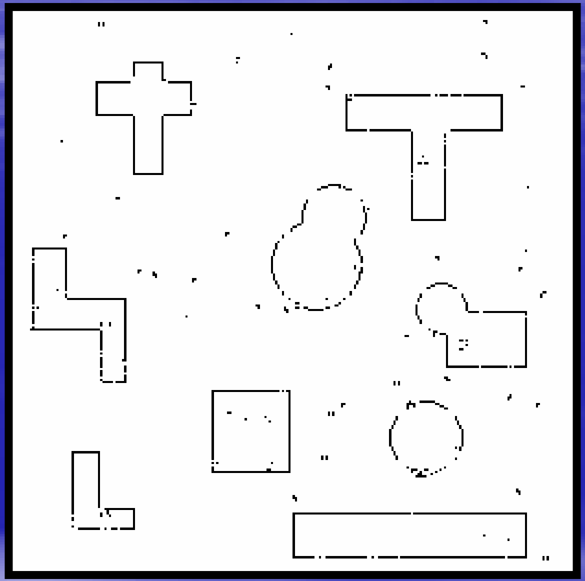
→ c) **Machine design after restoration**

ζ_{pac} designed by examples from restored images

Machine design over noisy images	Human design after restoration	Machine design after restoration
0.65 %	0.27 %	0.24 %

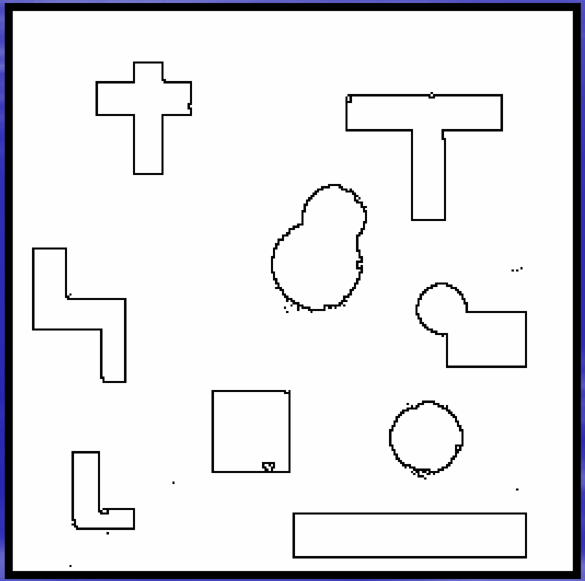


Noise Edge Detection



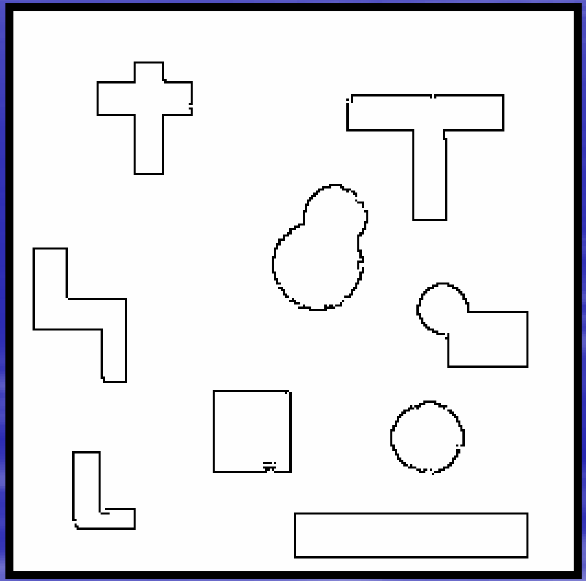
Machine design over noisy images

Error = 0.65%



Human design after restoration

Error = 0.27%



Machine design after restoration

Error = 0.24%

Envelope multi-resolution constraint

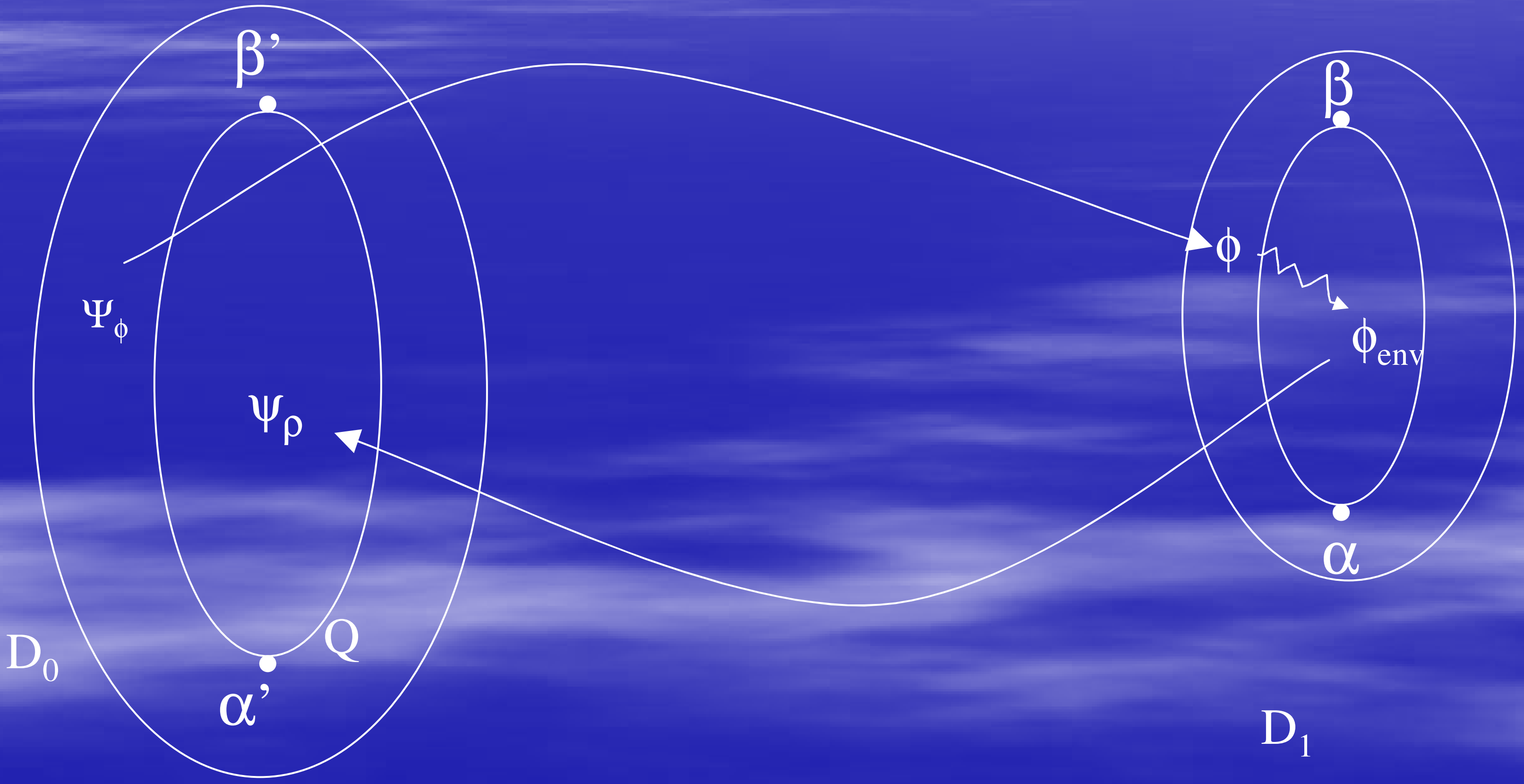
Definition

- $W_1 \subset W_0$, $\rho: D_0 \to D_1$ is a resolution mapping
- $\alpha, \beta: D_1 \to \{0,1\}$ with $\alpha \leq \beta$
- $\psi : D_0 \to \{0.1\}$

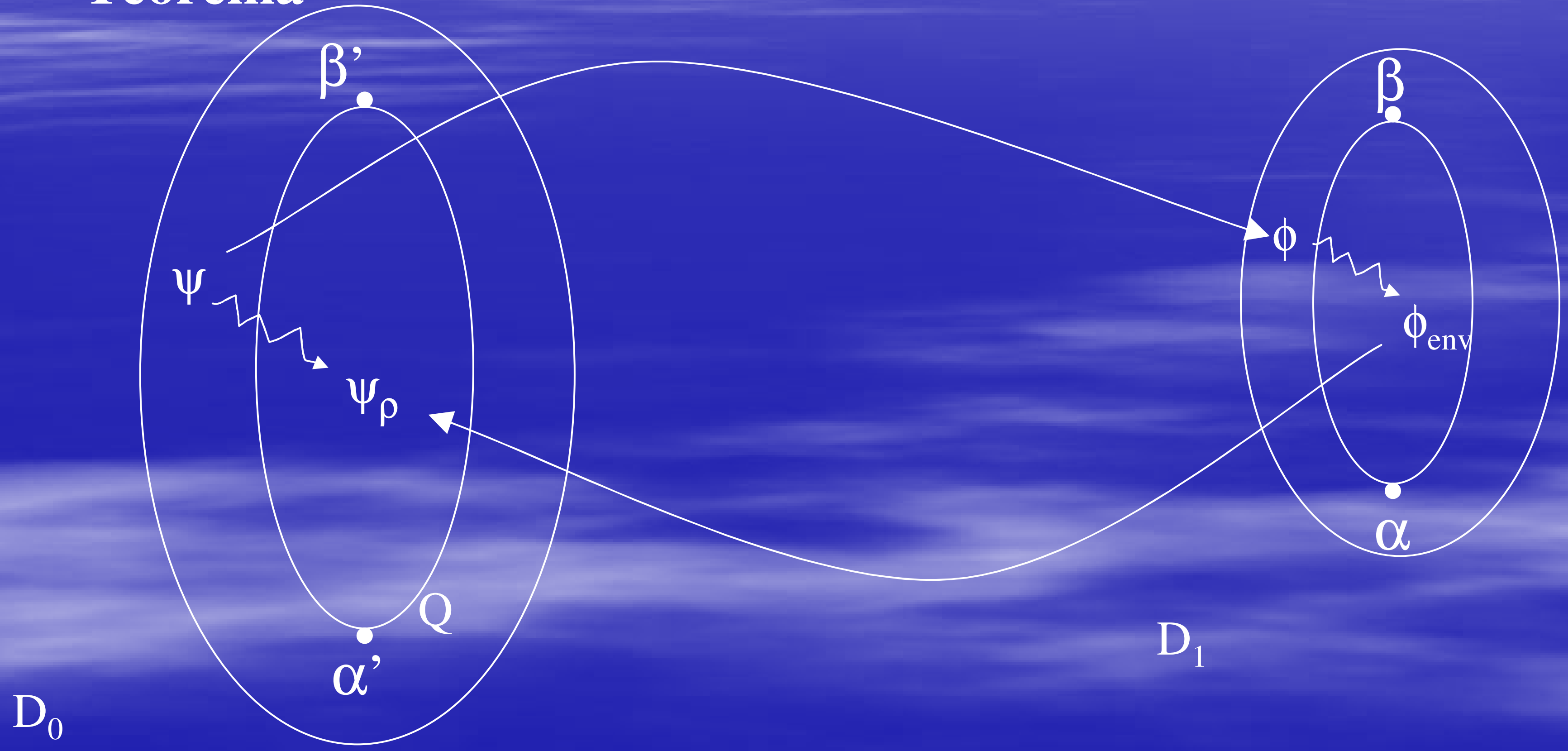
$$\psi_\rho(\mathbf{x}) = \begin{cases} 1 & if\, \alpha(\rho(x)) = 1 \\ 0 & if\, \beta(\rho(x)) = 0 \\ \psi(\mathbf{x}) & otherwhise \end{cases}$$

- $\psi_\rho = (\psi \wedge \beta') \vee \alpha'$, $\alpha'(x) = \alpha(\rho(x))$ and $\beta'(x) = \beta(\rho(x))$

Definition



Teorema



Piramidal Design:

Let $\psi_{i,env,N} = (\psi_{i,N} \wedge \beta') \vee \alpha'$, be the projection of the resolution constrained filter inside the envelope (α', β')

$$\psi_{env-mres}(\mathbf{x}) = \begin{cases} \psi_{0,N}(\mathbf{x}) & if \quad N(\mathbf{x}) > 0 \\ \psi_{1,env,N}(\mathbf{x}) & if \quad N(\mathbf{x}) = 0, N(\rho_1(\mathbf{x})) > 0 \\ \vdots & \\ \psi_{m-1,env,N}(\mathbf{x}) & if \quad N(\mathbf{x}) = 0, \ldots, N(\rho_{m-2}(\mathbf{x})) = 0, N(\rho_{m-1}(\mathbf{x})) > 0 \\ \psi_{m,env,N}(\mathbf{x}) & if \quad N(\mathbf{x}) = 0, \ldots, N(\rho_{m-1}(\mathbf{x})) = 0, N(\rho_m(\mathbf{x})) > 0 \end{cases}$$

Properties:

- $\psi_{\text{env-mres}}$ is a consistent estimator of ψ_{opt}

- If the envelope is well defined on D_1, then the ρ-envelope of a resolution constrained filter is advantageous

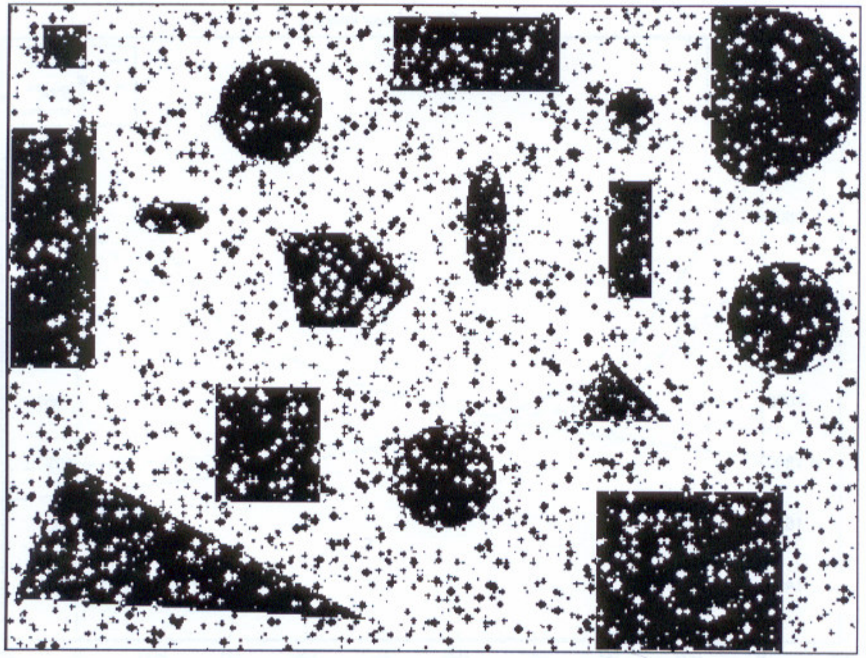
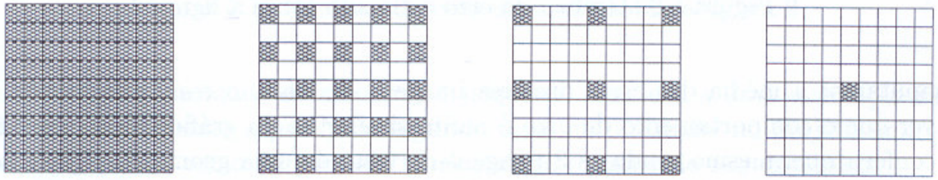
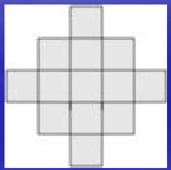


Figura 5.7: Primeira imagem corrompida com ruído

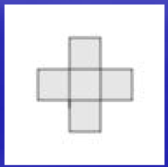


(a) (b) (c) (d)

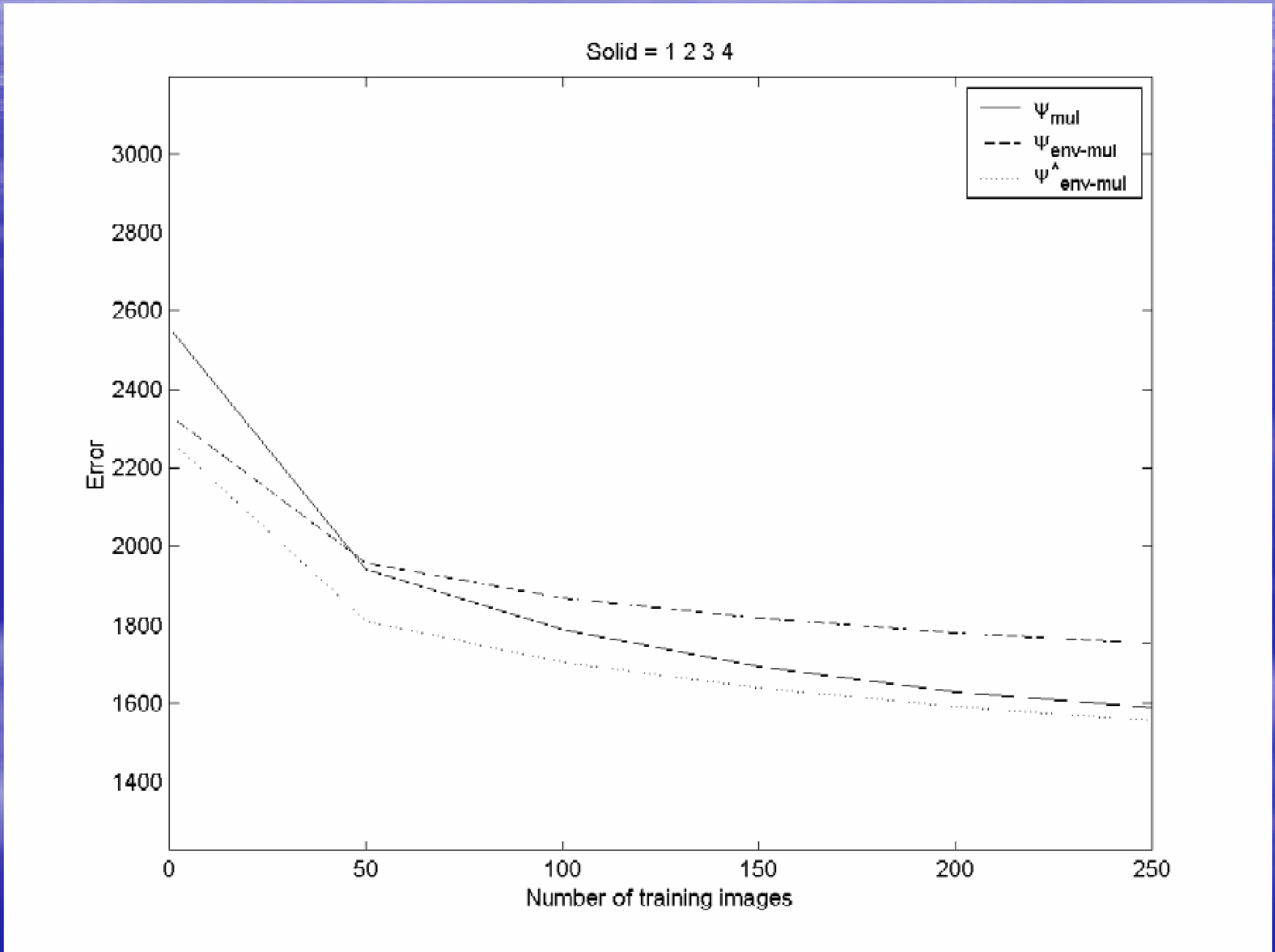
B



E

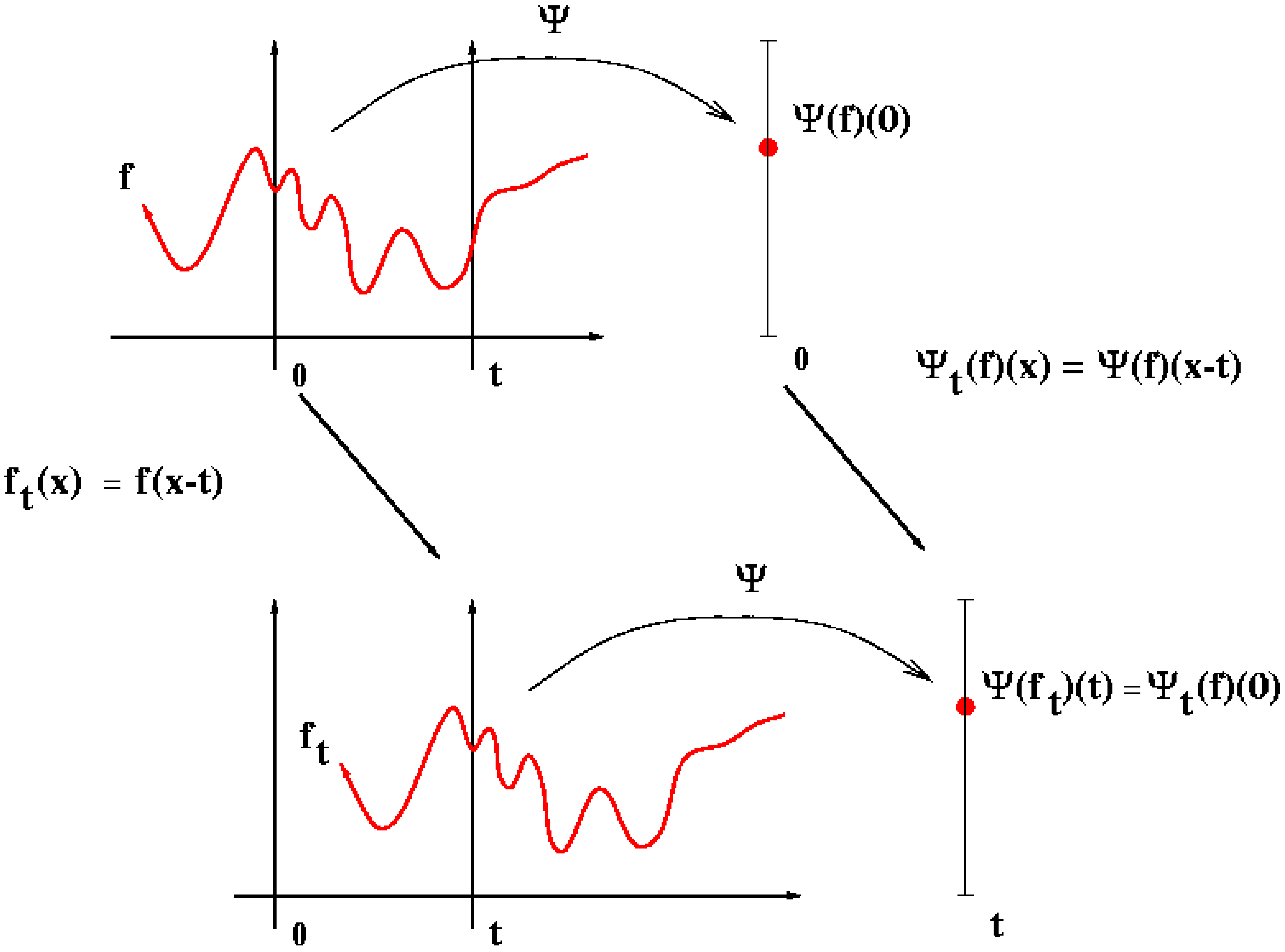


$$\alpha = \varepsilon_B(\gamma_E \phi_E \gamma_E)$$
$$\beta = \delta_B(\phi_E \gamma_E \phi_E)$$

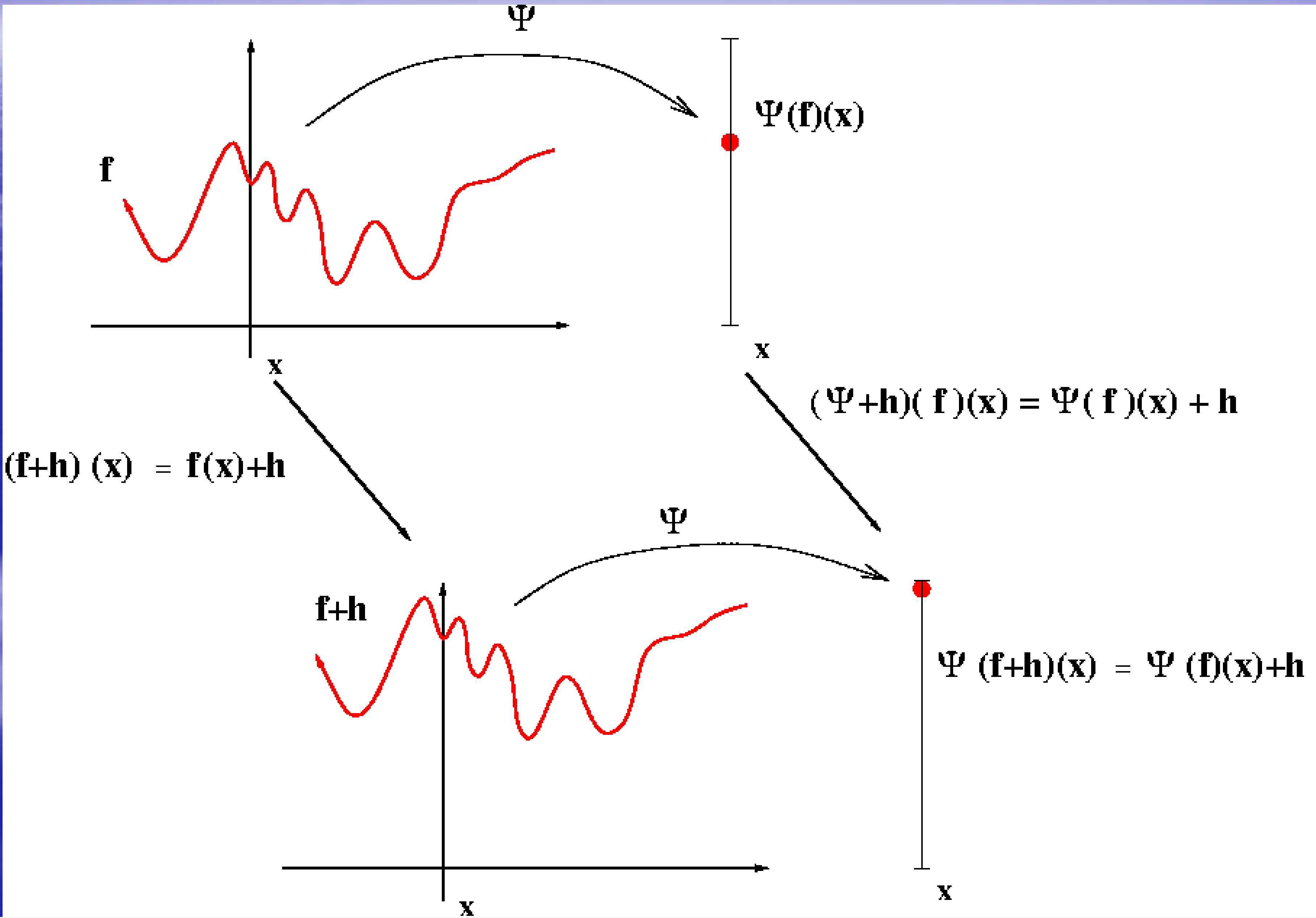

Solid = 1 2 3 4
3000
2800
2600
2400
2200
2000
1800
1600
1400
Error
0
50
100
150
200
250
Number of training images
Ψmul
Ψenv-mul
Ψ^env-mul

Gray-scale operator design: aperture

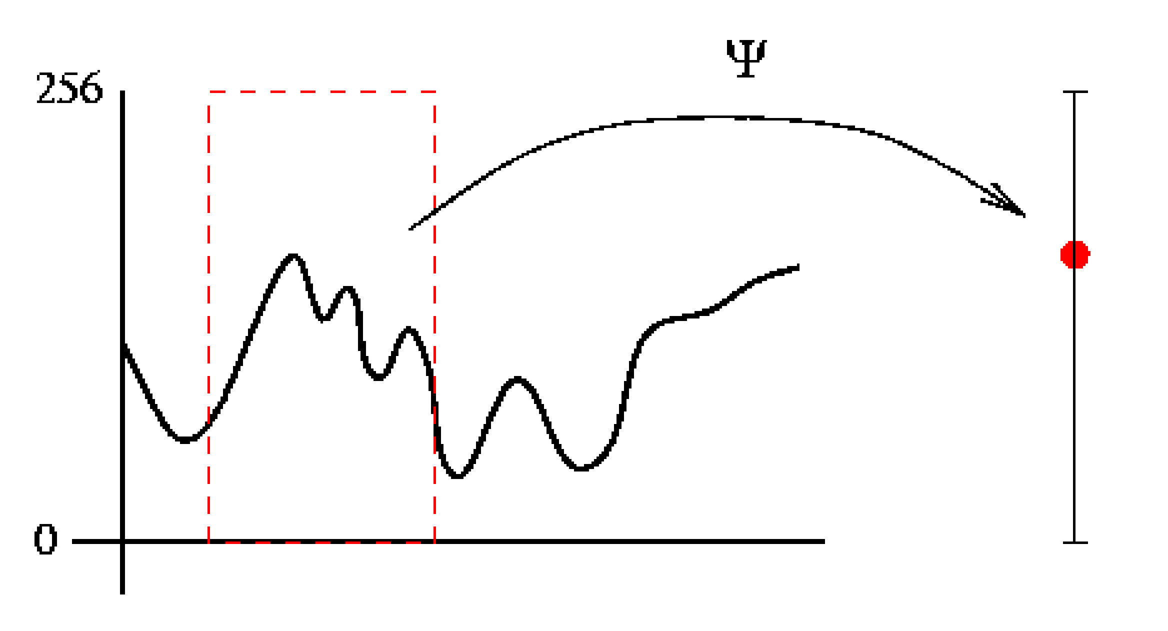
Spatial Translation Invariance



Gray-scale Translation Invariance

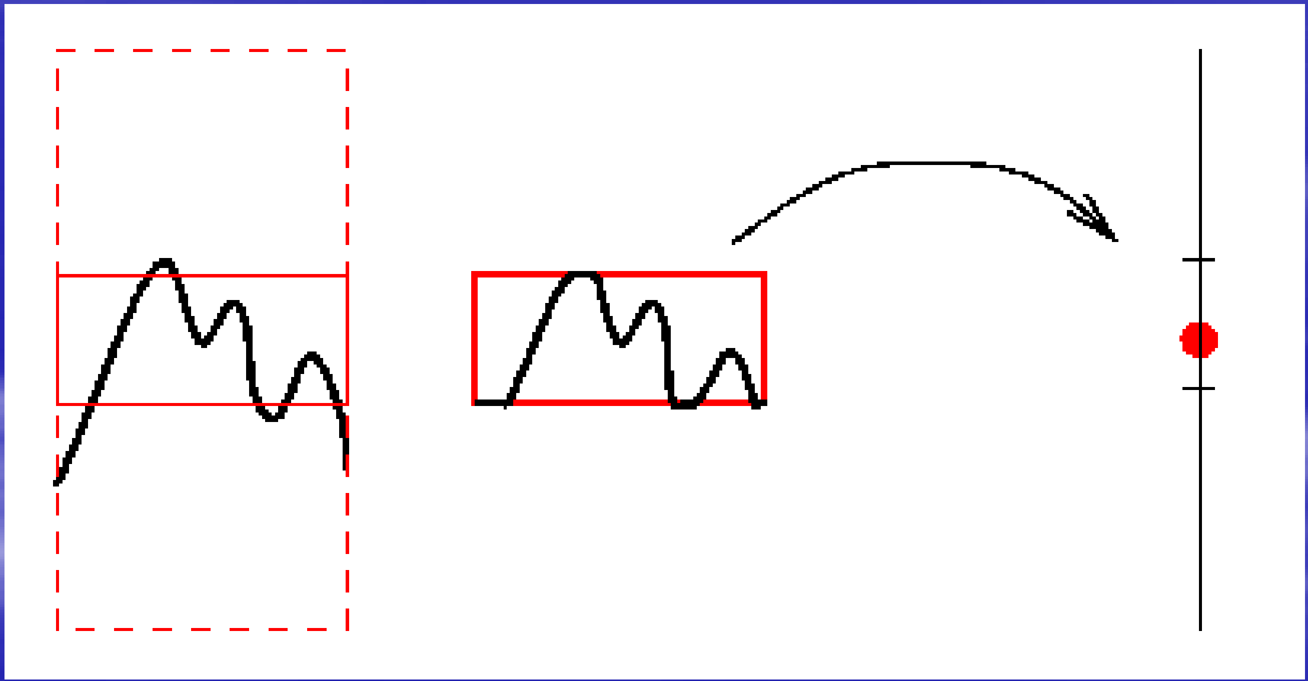


Locally defined in W



$$\Psi(f)(x) = \Psi(f / W_x)(x)$$

Locally defined in W and K



$$(u/K_y)(z) = \wedge\{\vee\{-k, u(z)-y\}, k\}$$

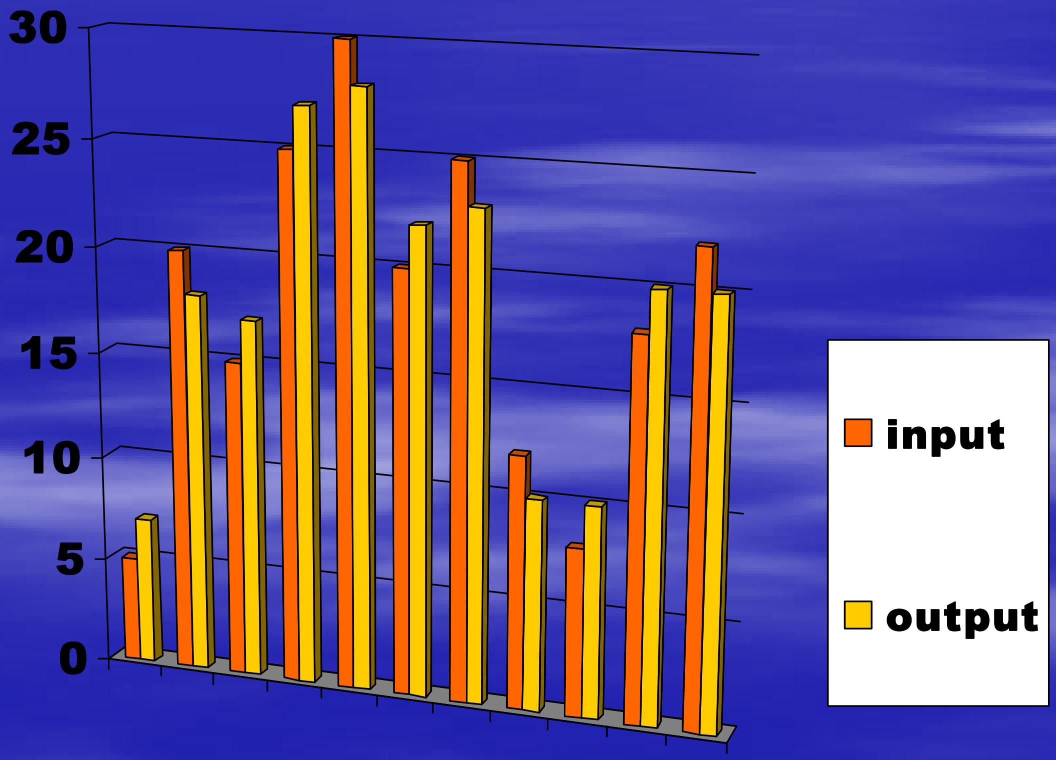
Aperture Operator

W

$$K = \{-2, -1, 0, 1, 2\}$$

β_ψ

2	-2	1	2	2	2
1	-2	1	2	2	2
0	-2	1	2	2	2
-1	-2	1	1	1	1
-2	-2	-2	-2	-2	-2
	-2	-1	0	1	2



Aperture Operator

β_ψ

	-2	-1	0	1	2
2	-2	1	2	2	2
1	-2	1	2	2	2
0	-2	1	2	2	2
-1	-2	1	1	1	1
-2	-2	-2	-2	-2	-2

ψ = $u(o)$ + β_ψ

ψ:

	10	11	12	13	14
14	12	13	14	15	16
13	12	13	14	15	15
12	12	13	14	14	12
11	12	13	13	11	12
10	12	12	10	11	12

=

$u(o)$:

	10	11	12	13	14
14	10	11	12	13	14
13	10	11	12	13	14
12	10	11	12	13	14
11	10	11	12	13	14
10	10	11	12	13	14

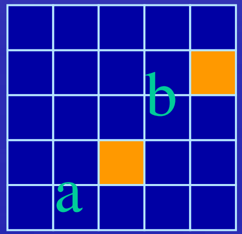
+

β_ψ:

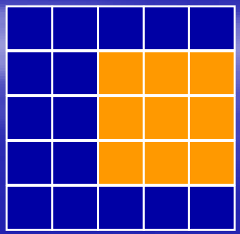
	10	11	12	13	14
14	2	2	2	2	2
13	2	2	2	2	1
12	2	2	2	1	-2
11	2	2	1	-2	-2
10	2	1	-2	-2	-2

- Let $a, b \in$ Fun[W,L], $a \leq b$ iff $a(x) \leq b(x)$, $x \in W$

|W| = 2

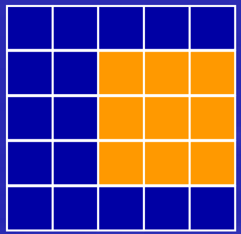


- Interval $[a,b] = \{u \in$ Fun$[W,L]$: $a \leq u \leq b\}$



- Sup-generating operator:

$$\lambda_{a,b}(u) = 1 \Leftrightarrow u \in [a,b]$$



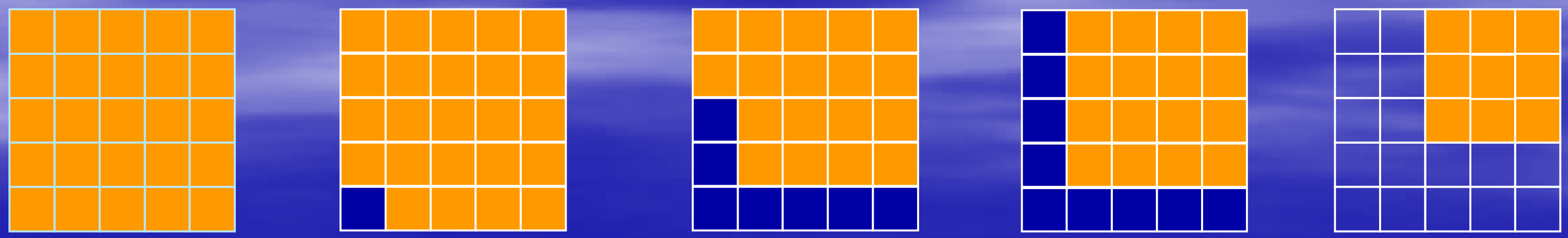
[a,b]

0	0	0	0	0
0	0	1	1	1
0	0	1	1	1
0	0	1	1	1
0	0	0	0	0

$\lambda_{a,b}$

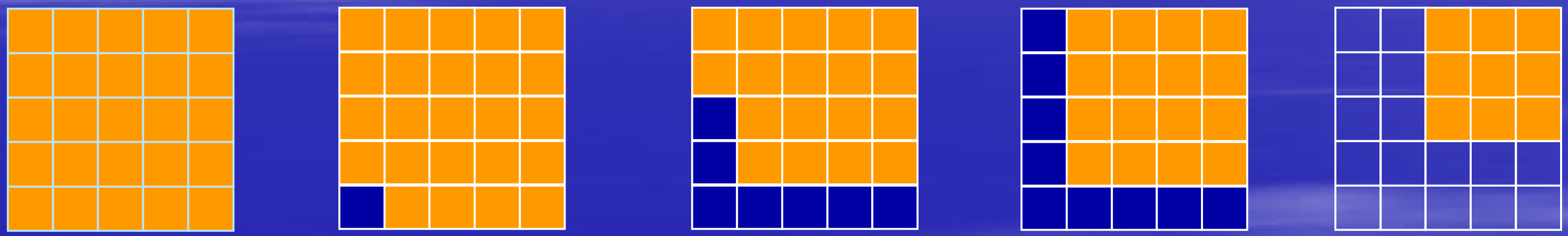
Kernel of ψ at y: $K(\psi)(y) = \{u \in \mathrm{Fun}[W,L] : y \leq \psi(u)\}$

2	0	1	2	2	2
1	0	1	2	2	2
0	-1	1	2	2	2
-1	-1	1	1	1	1
-2	-2	-1	-1	-1	-1
	-2	-1	0	1	2

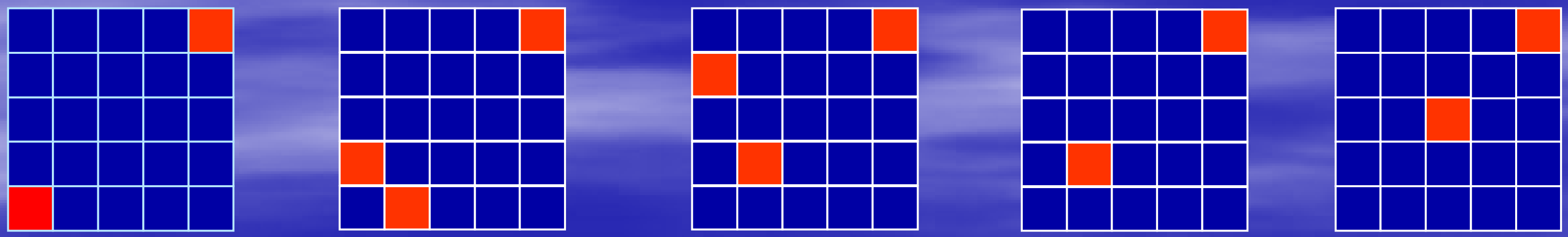


K(ψ)(-2)   K(ψ)(-1)   K(ψ)(0)   K(ψ)(1)   K(ψ)(2)

Basis of ψ at *y*: B(ψ) is the set of maximal intervals contained in K(ψ)



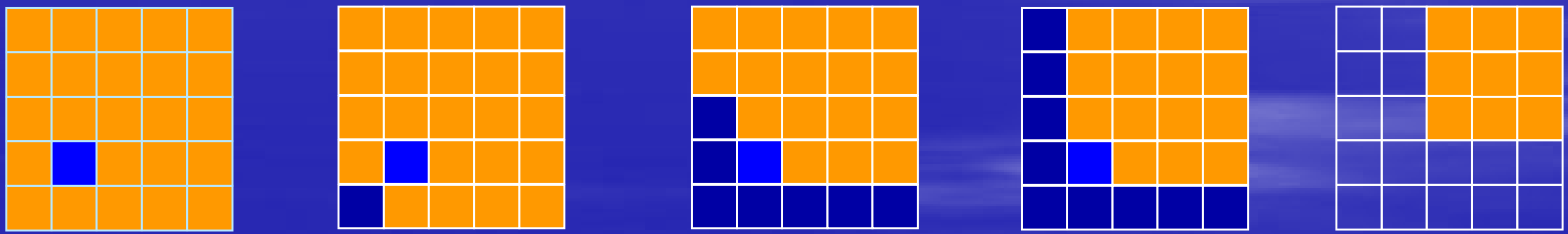
K(ψ)(-2) K(ψ)(-1) K(ψ)(0) K(ψ)(1) K(ψ)(2)



B(ψ)(-2) B(ψ)(-1) B(ψ)(0) B(ψ)(1) B(ψ)(2)

Sup-representation

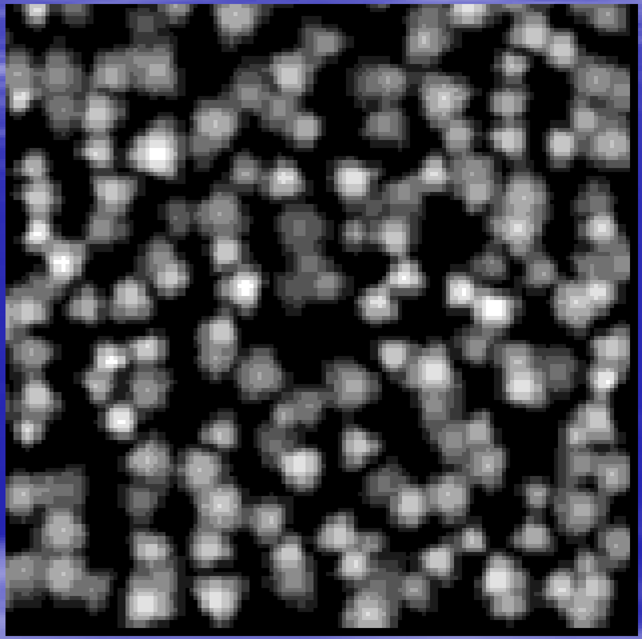
$$\psi(u) = \bigcup\left\{ y \in M : \bigcup\left\{ \lambda_{a,b}(u) : [a,b] \in B(\psi)(y) \right\} = 1 \right\}$$



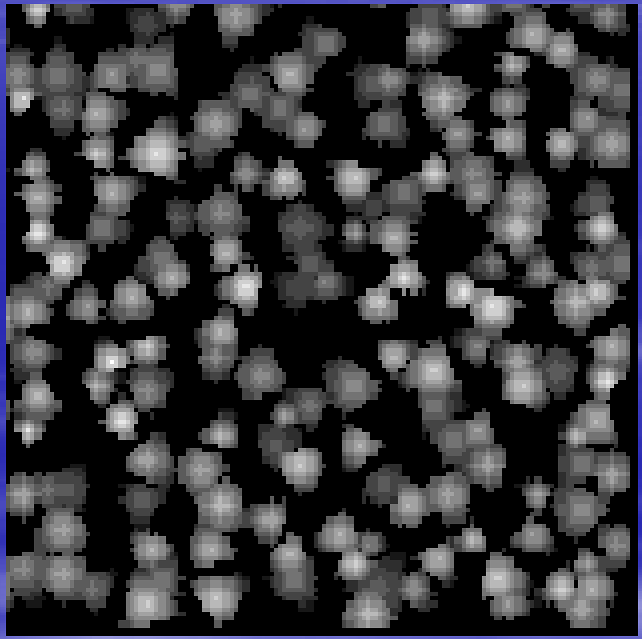
K(ψ)(-2) K(ψ)(-1) K(ψ)(0) K(ψ)(1) K(ψ)(2)

ψ(-1,-1) = 1

Observed

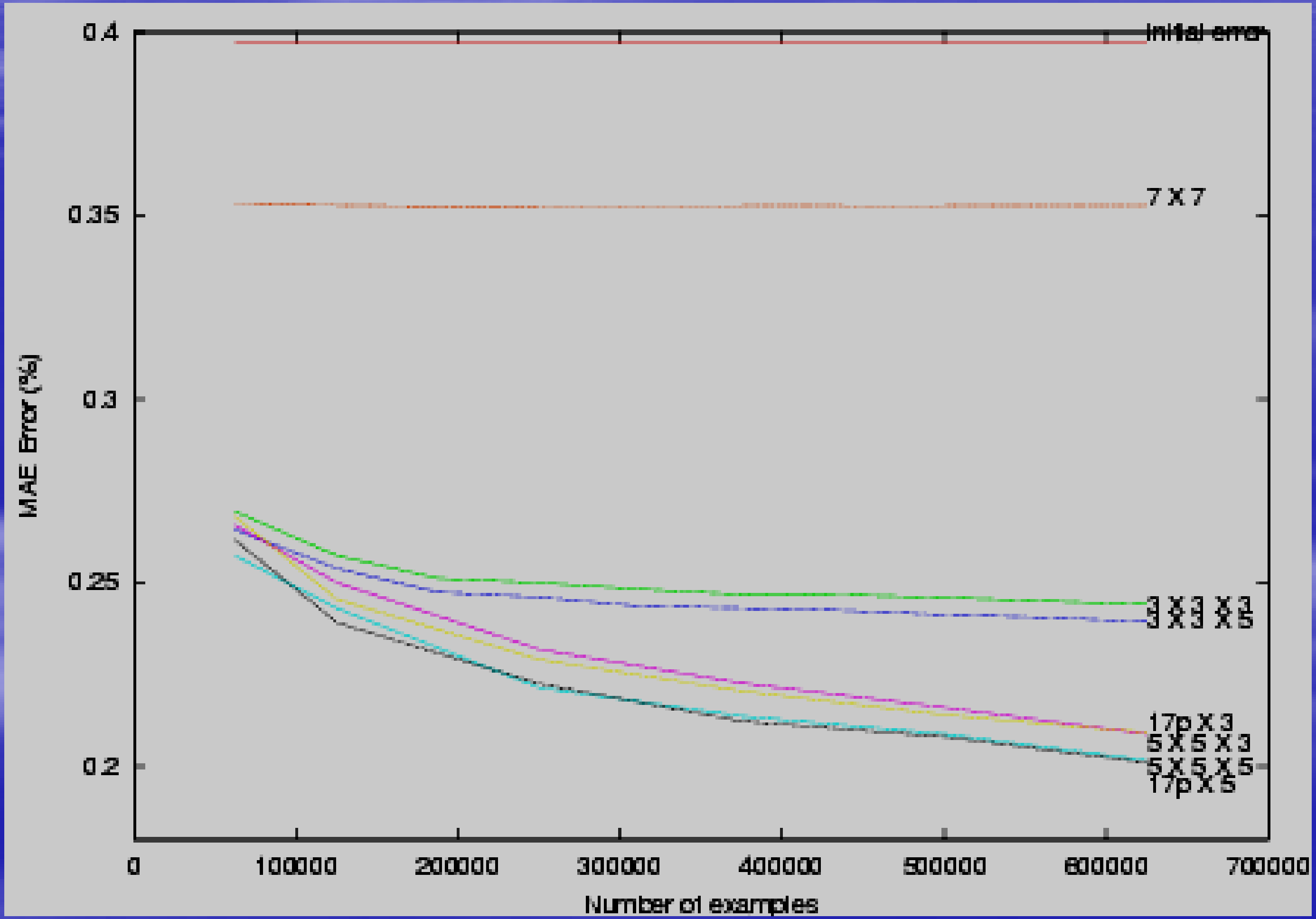


Ideal



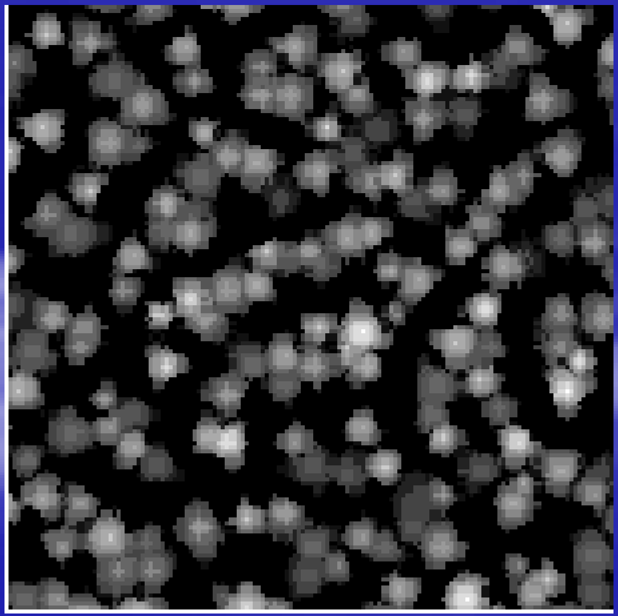
These are part of the observed and ideal images (512x512)

MAE x Number of Examples

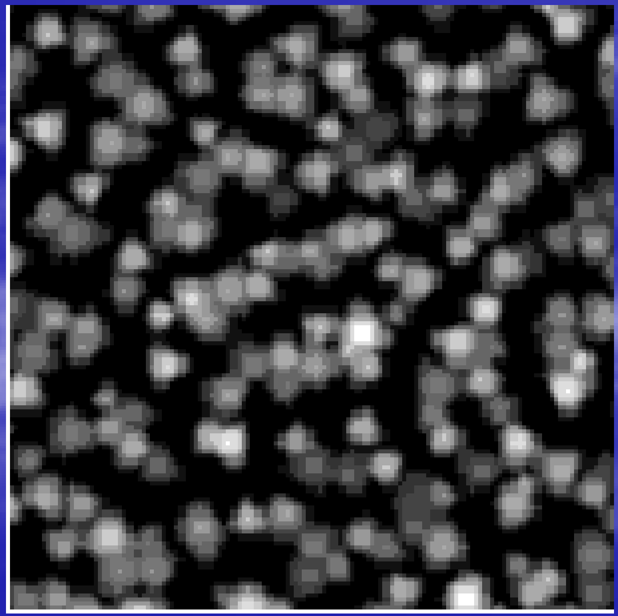


Debluring - Aperture x Optimal linear

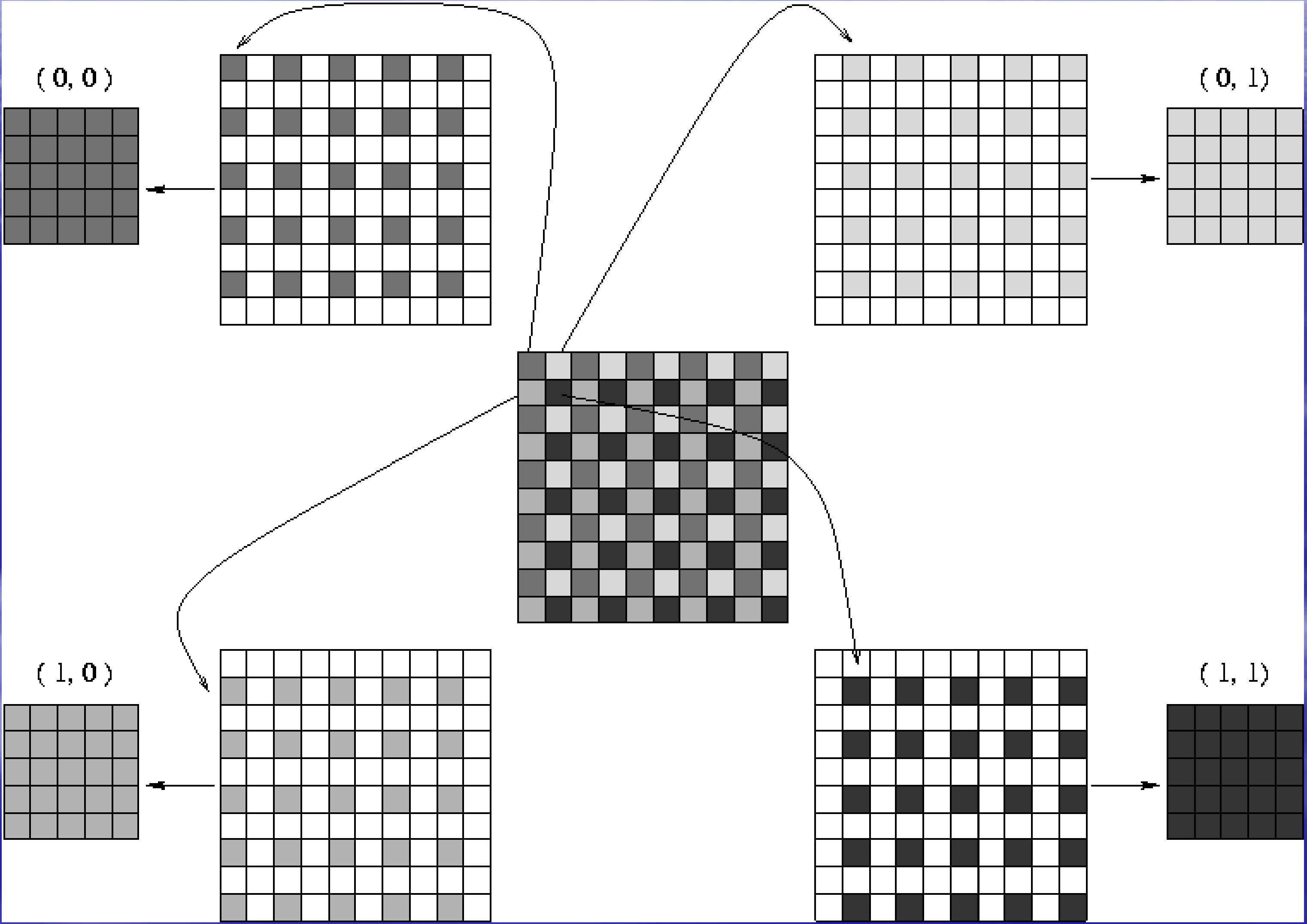
Aperture 17p x 5 x 5



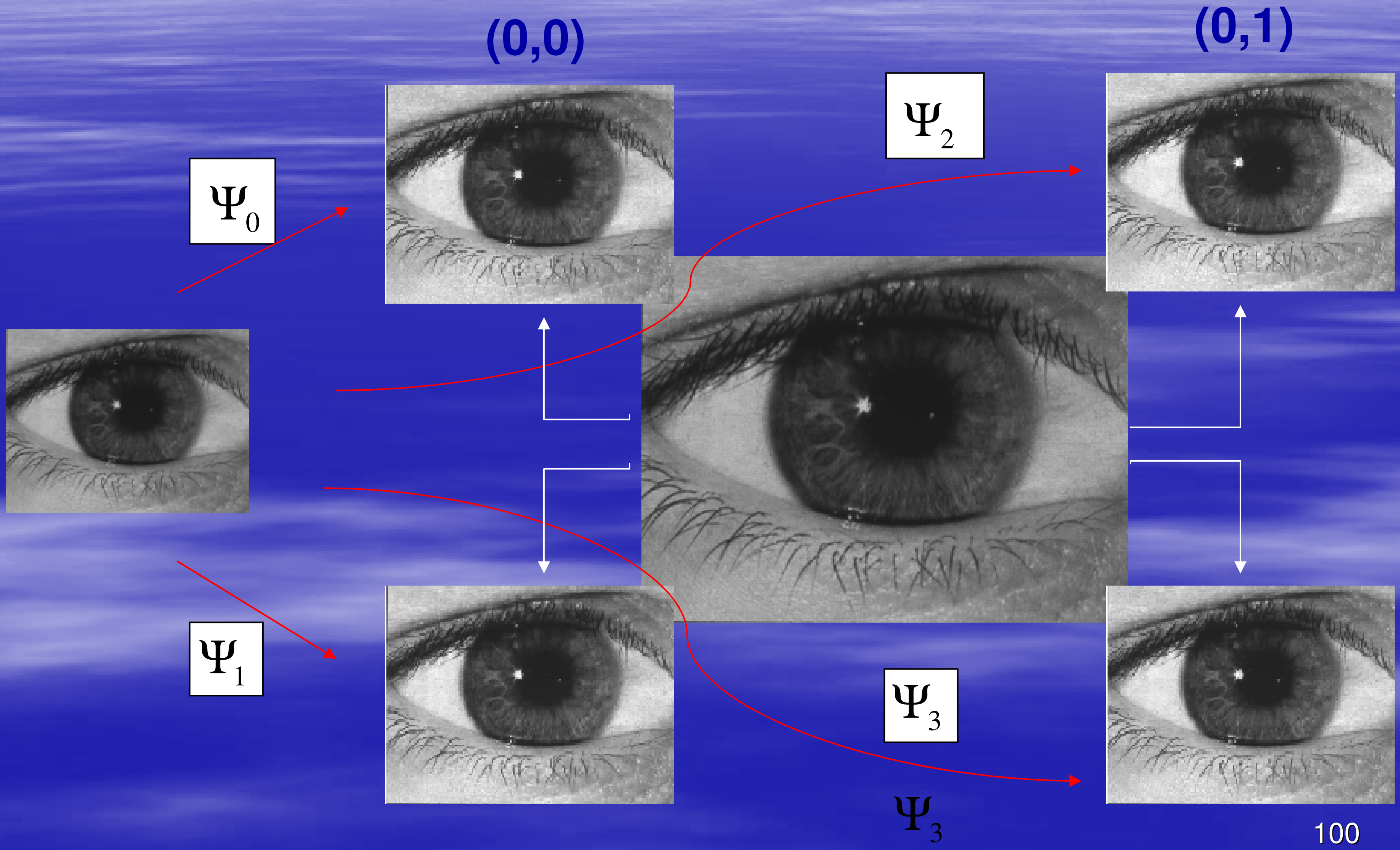
Optimal linear 7x7



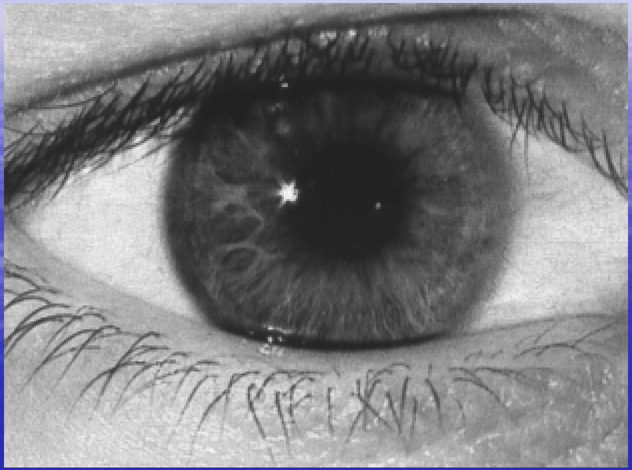
Resolution Enhancement



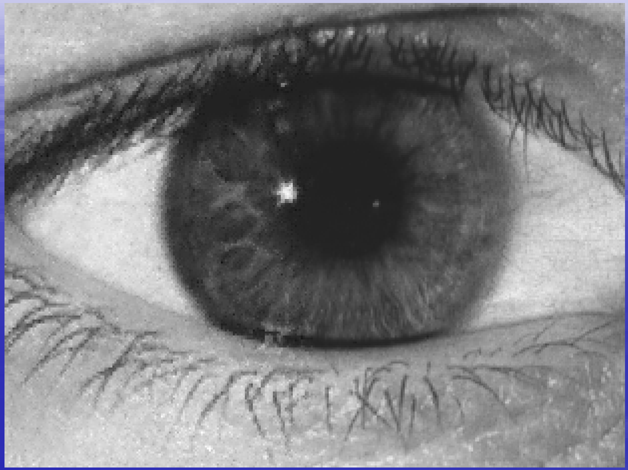
Resolution Enhancement



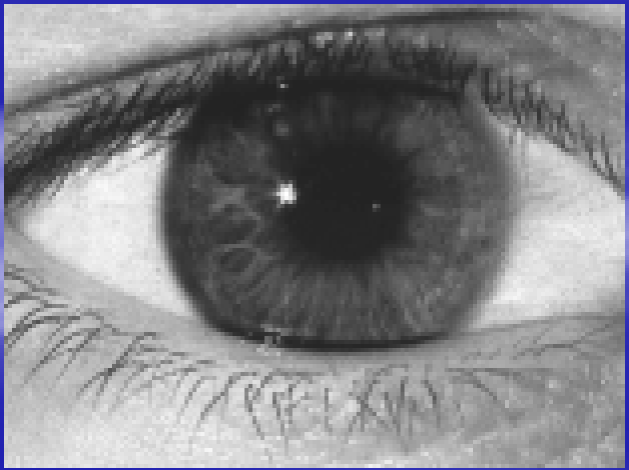
Resolution Enhancement - Results



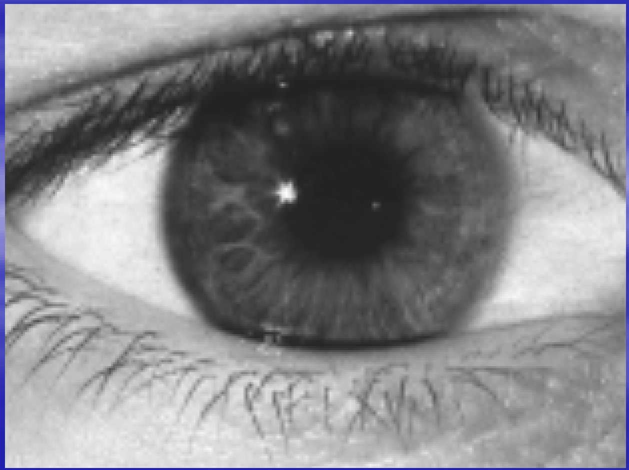
Original



Aperture: 3x3x21x51



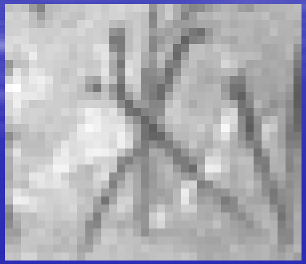
Linear



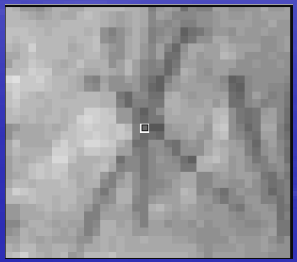
Bilinear

Resolution Enhancement - Results

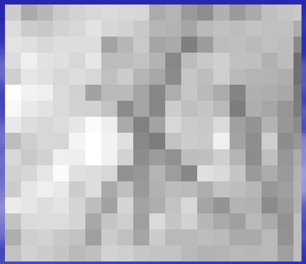
Zoom



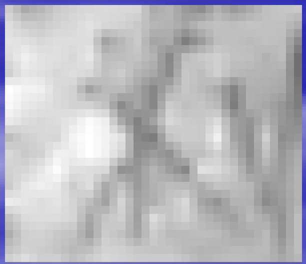
Original



Aperture: 3x3x21x51



Linear



Bilinear

Gray-scale operator design: stack filters

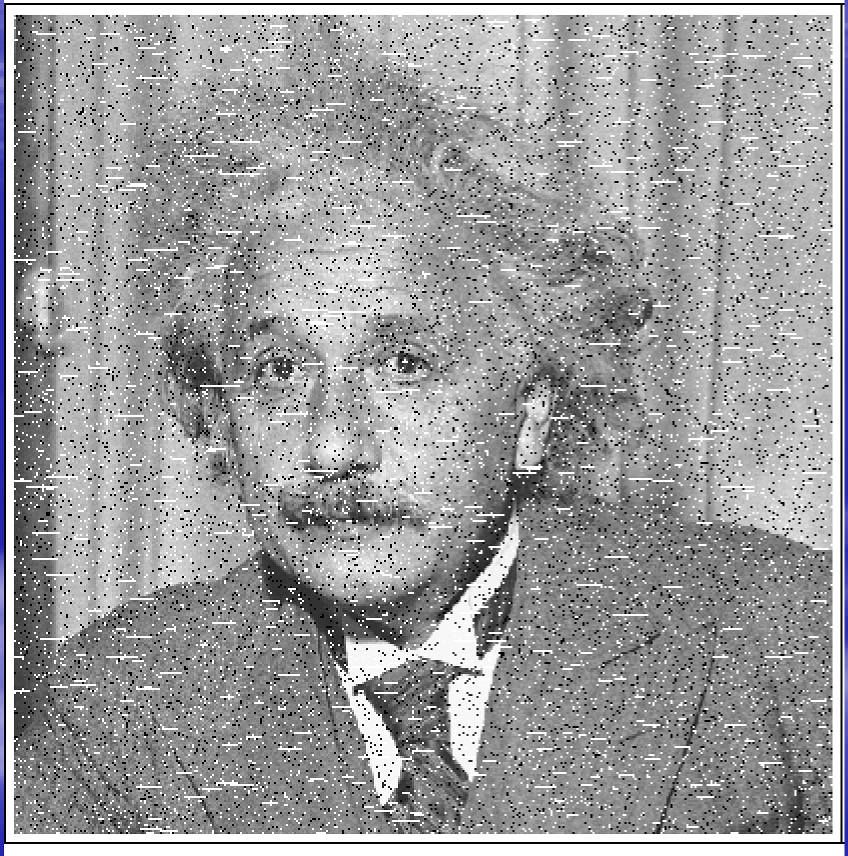
A stack filter is a gray-scale operator characterized by a positive (i.e., increasing) Boolean function

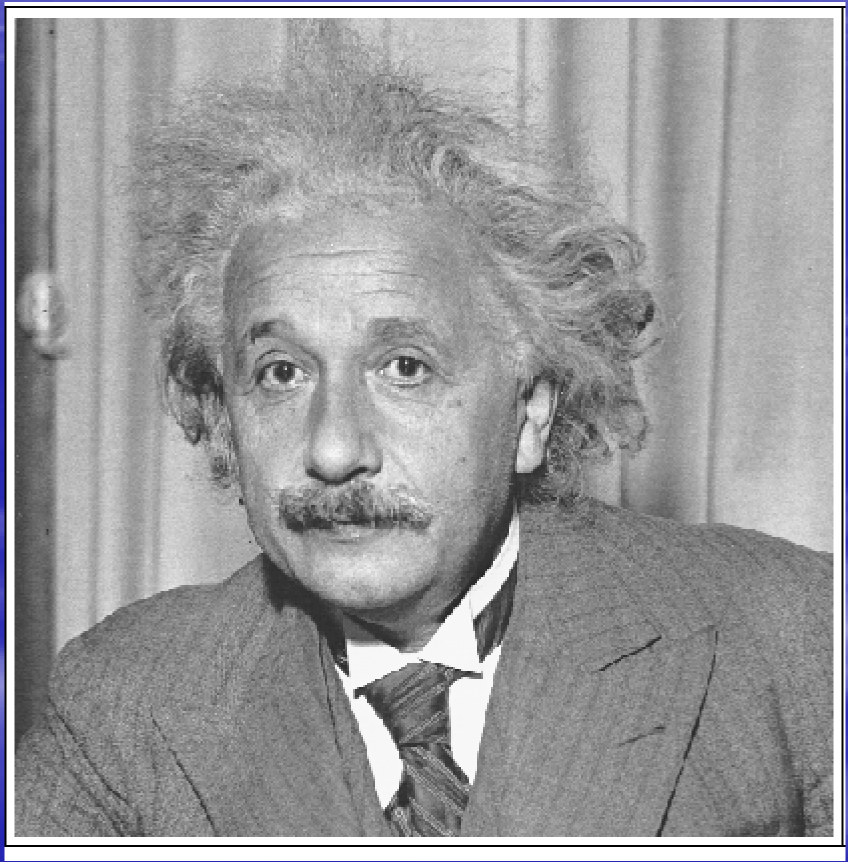
$$\psi(f) = \max\{t \in K : \psi(T_t[f]) = 1\}$$

where

$$T_t[f] = \{x \in W : f(x) \geq t\}$$

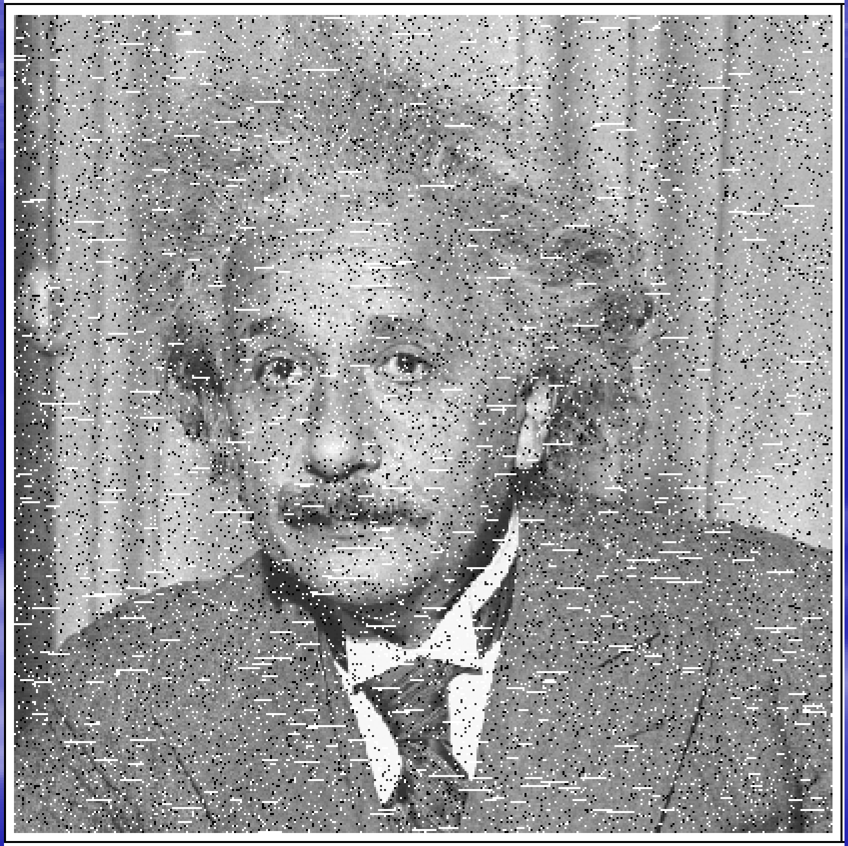
Impulse noise removal (1)



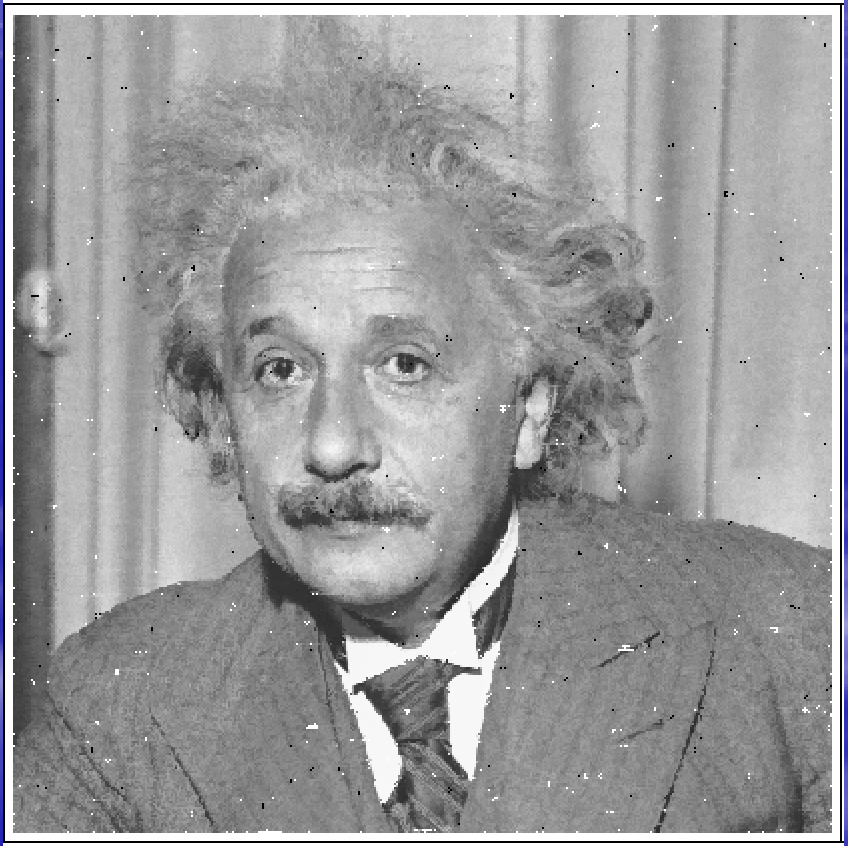


training images

Impulse noise removal (2)

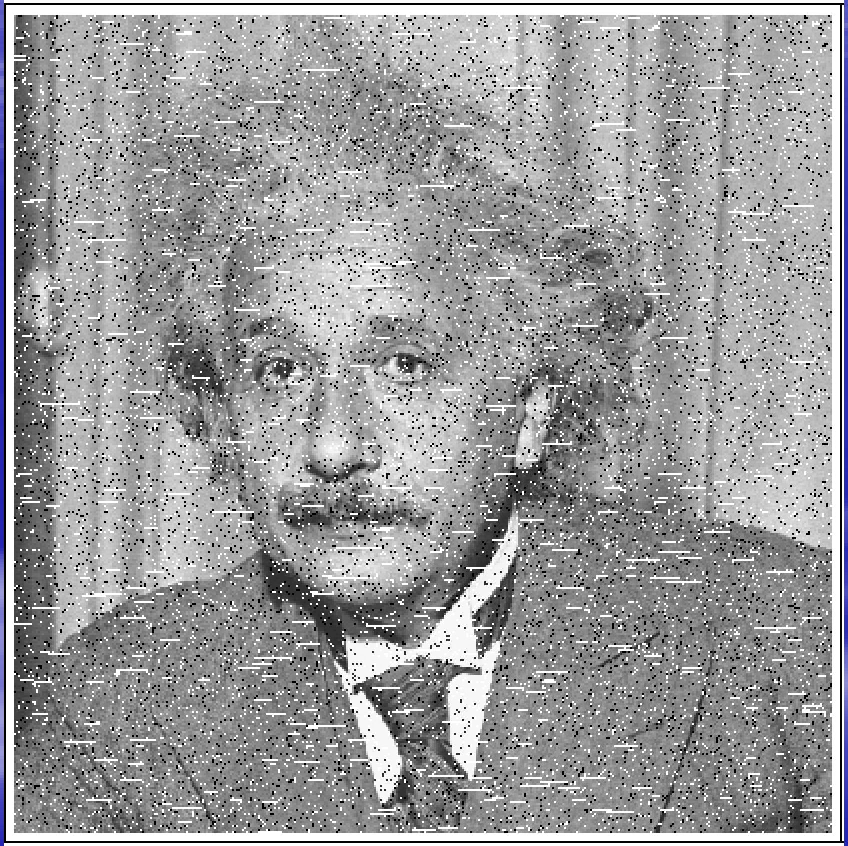


test image

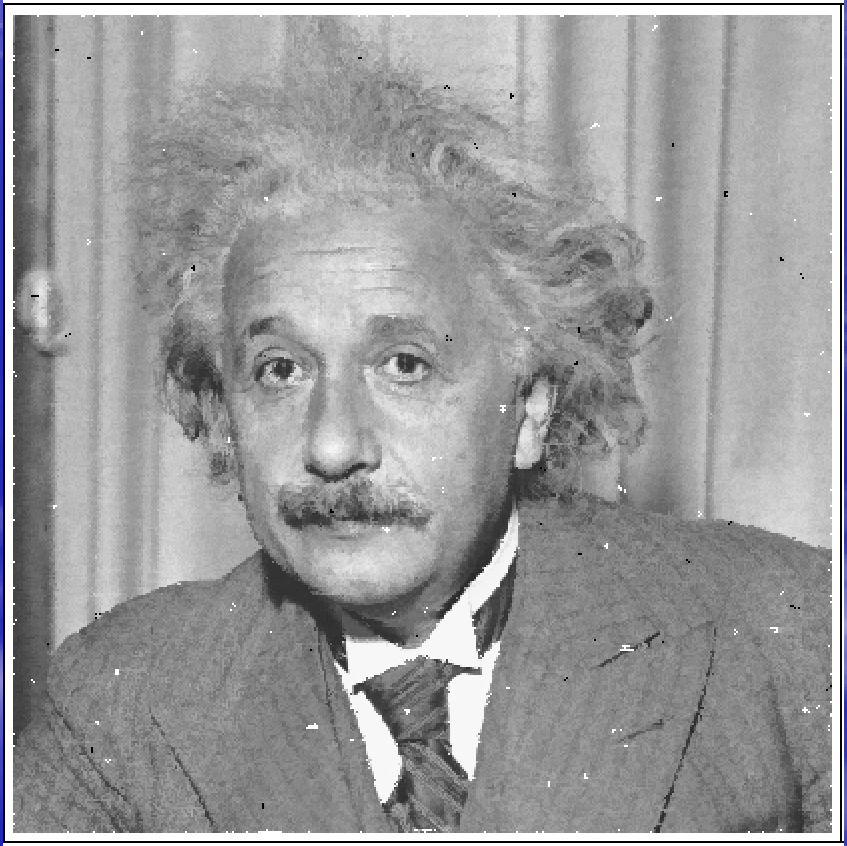


iteration 1

Impulse noise removal (3)



test image



iteration 5

Robustness (1)

test image

iteration 1

Robustness (2)

test image

iteration 5

Conclusion

- Design of Morphological Operators: a discrete nature problem
- Fundamentals: Algebra, Statistics, Combinatory
- Real problems solution
- Design techniques adequate to introduce prior knowledge
- Identification of Lattice Dynamical Systems