Model-Based Design of Optimal Morphological Filters

Edward R. Dougherty

Department of Electrical Engineering

Texas A&M University

Junior Barrera

Institute of Mathematics and Statistics

University of Sao Paulo

Layout

- Introduction
- Optimal non-linear filter design
- Optimal linear filter design
- Connected openings and Granulometries
- Optimal granulometric filters
- Conclusion

Introduction

- A fundamental problem in image processing is the design of filters
- Filters may be learned from pairs of images: (input, expected output)
- Let us show how to learn connected bandpass filters
- Connected filters do not introduce new edges in the image
- Connected bandpass filters are sieving filters.

Optimal non linear filter design

Problem Formulation

- Images are random sets: input X, ideal S
- A filter Ψ is a set mapping
- Mean Absolute Error: $MAE(\Psi) = E[|\Psi(X) \Delta S|]$
- Ψ is in a family of filters F
- Representation: $\Psi(X) = \bigcup \{\lambda_{[A,B]}(X) : [A,B] \subseteq B(\Psi)\}$
- Ψ_{opt} is such that $MAE(\Psi_{opt}) \leq MAE(\Psi)$ for all Ψ in F

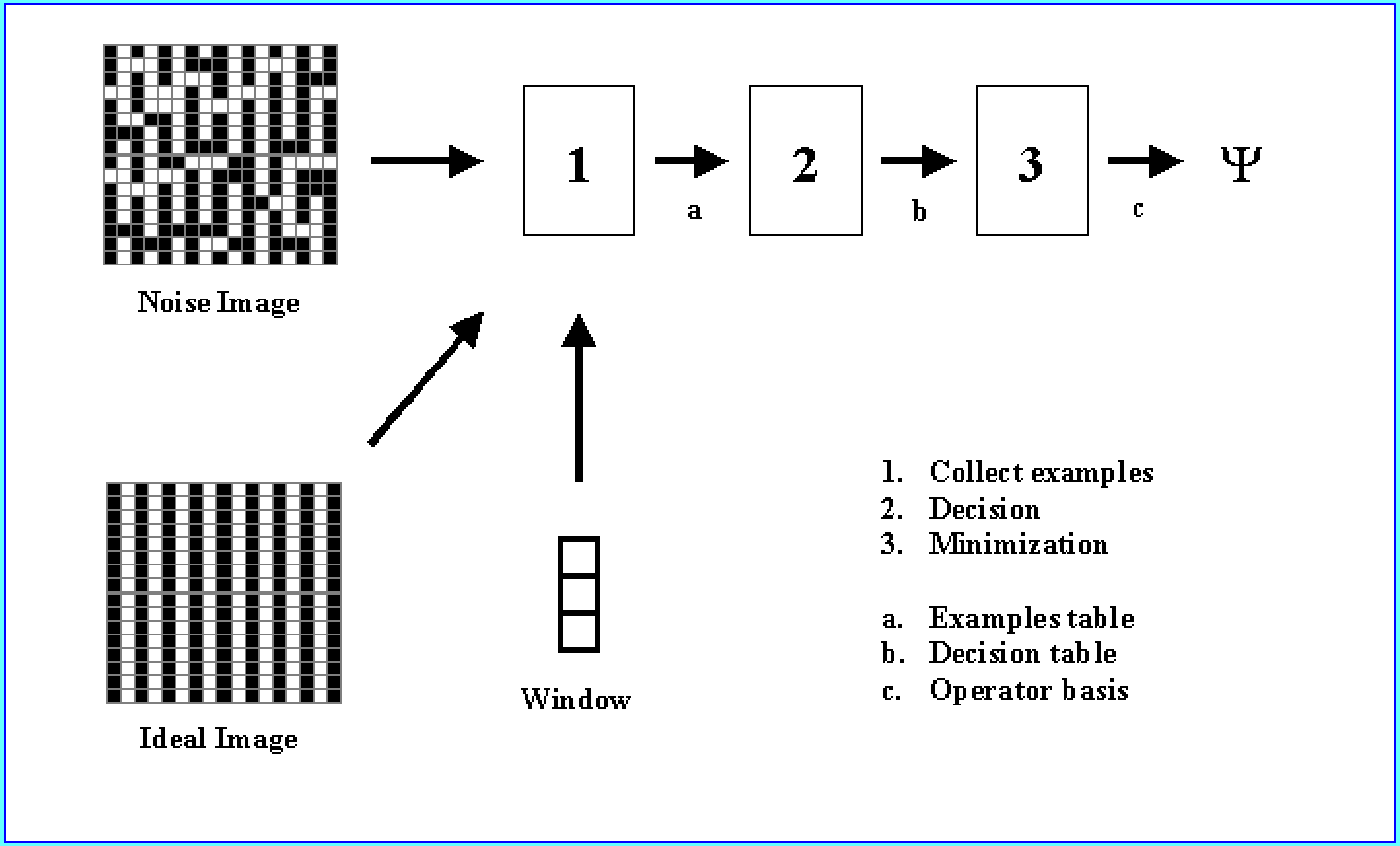
Difficulty

There is no closed formula for Ψ_{opt} in terms of statistics of X and S.

Approach

- Estimation of the joint probability distribution between observations and the target variable to be estimated.
- Optimization via search over filter space
- Logic reduction to find a minimal filter representation.

Complete Scheme



Collect and Decision

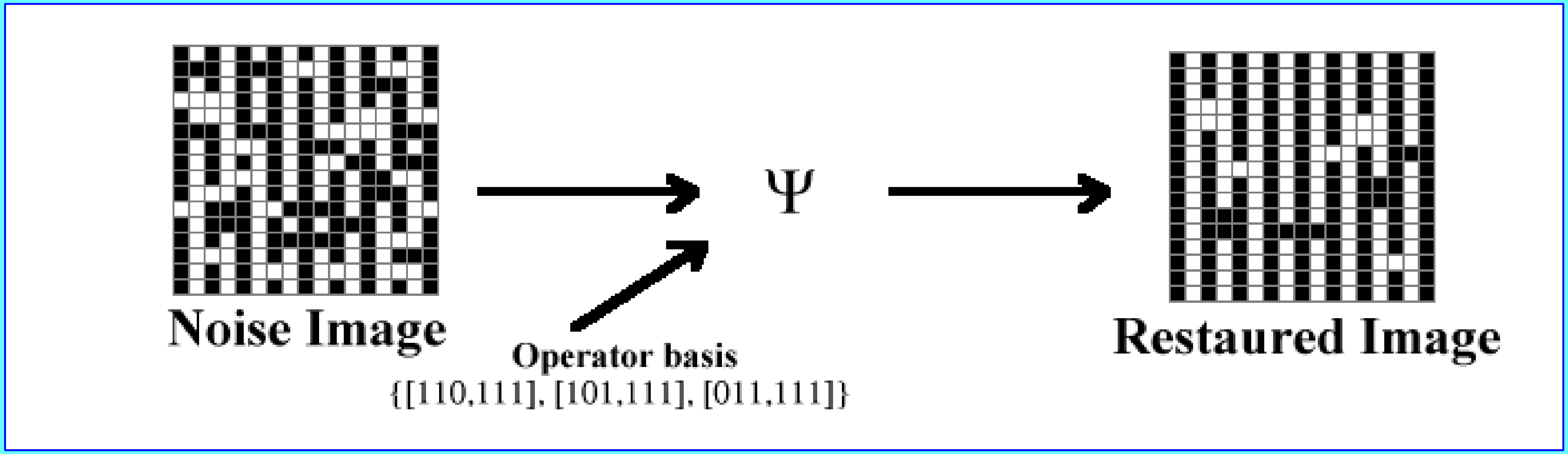
$x_1x_2x_3$	Frequency of 0	Frequency of 1
0 0 0	86	0
0 0 1	19	2
0 1 0	18	0
0 1 1	1	16
1 0 0	19	2
1 0 1	0	14
1 1 0	1	16
1 1 1	0	78

a

$x_1x_2x_3$	h(x)
0 0 0	0
0 0 1	0
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	1
1 1 1	1

b

Application



Difficulty

Large windows implies in lack of data, that is, serious estimation errors

Approach

Introduce prior information, that is,
constrain the family of filters

Optimal linear filter design

Linear Estimator

Estimate ideal signal $S(s)$ via an observed (zero-mean) signal $X(t)$ by linear operator

$$\Psi(X)(s) = \int_T g(s,t)X(t)dt$$

Optimization involves finding g(s,t) to minimize the Mean Square Error (MSE)

Wiener-Hopf equation

$g_{opt}(s,t)$ yields the optimal MSE linear estimator of S based on X(t) iff it satisfies the Wiener-Hopf equation,

$$R_{SX}(s,t) = \int_T g_{opt}(s,u) R_X(u,t) du$$

where R_X and R_{SX} are the auto-correlation and cross-correlation for X, and X and S

Power Spectral Density (PSD)

The optimal function $g_{opt}(s,t)$ is given by

$$g_{opt}(s,t) = \Phi^{-1}\left[\frac{H_{SX}(\omega)}{H_X(\omega)}\right]$$

where H_X and H_{SX} denote the PSD of R_X and R_{SX} ,
and Φ is the Fourier transform

Interpretation

Linear Filter: spectral decomposition; choice of bands; reconstruction

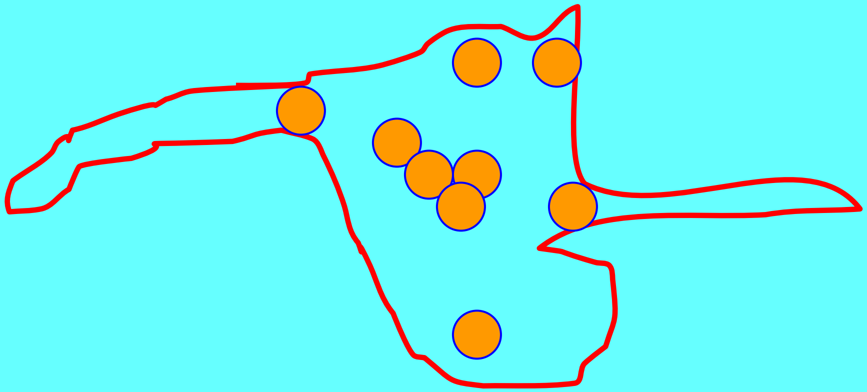
The analytic derivation of the optimal linear filter depends on the signal and filter representation, that is, spectral decomposition and convolution.

Connected openings and granulometries

Morphological Opening

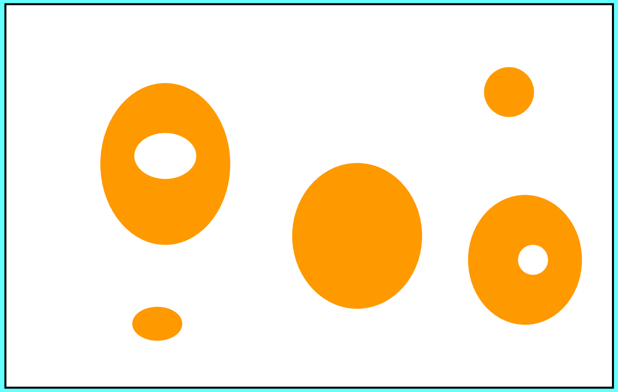
$$X \circ B = \bigcup \{B + x : B + x \subseteq X\}$$

where B+x is the translation of B by x

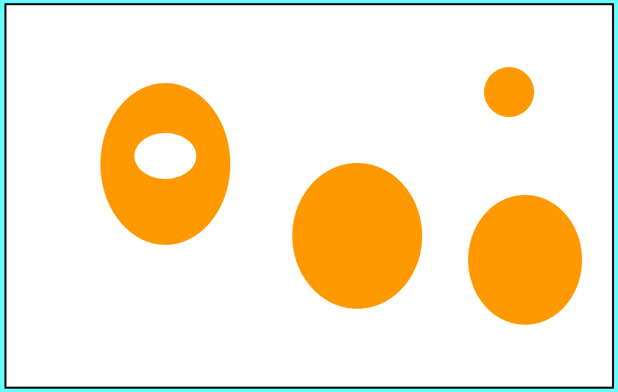


Connected Filter

Eliminate objects or holes of the image. Does not create new edges.



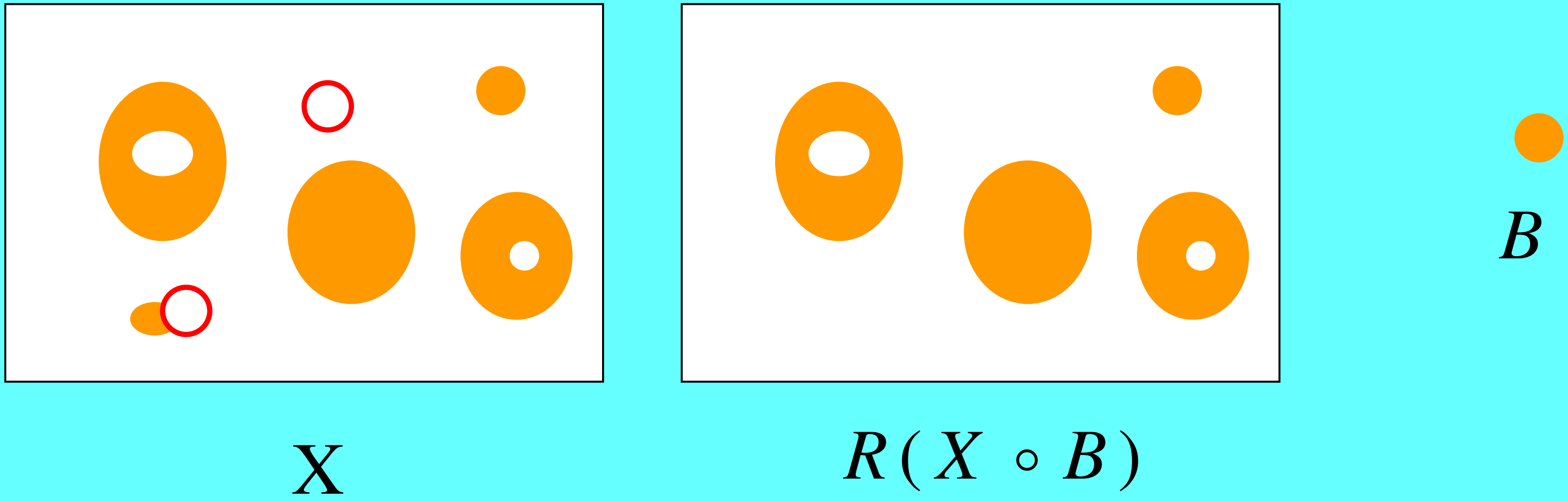
X

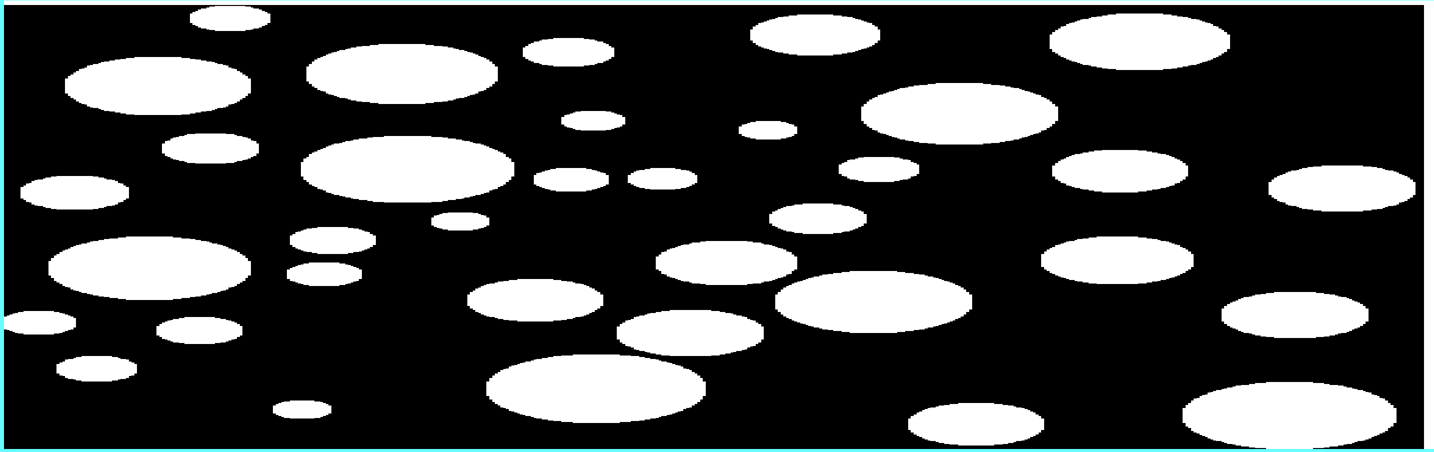


$\Psi (X)$

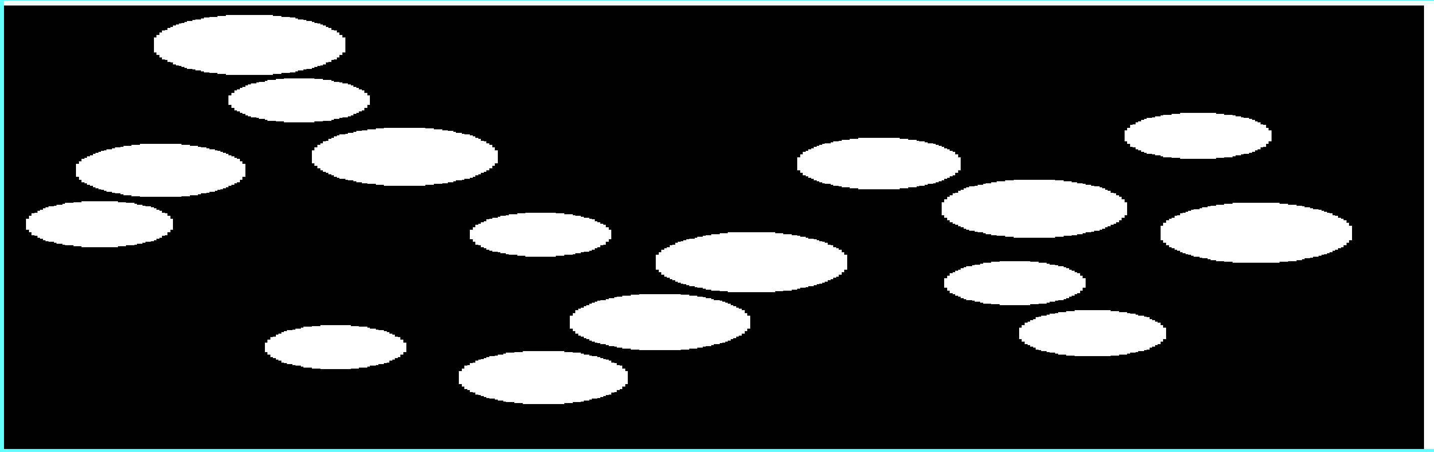
Single connected Opening

Eliminate objects for which does not exist x such that $B + x \subseteq X$





Original Image

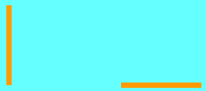


Opening $\Psi_t(X)$ by a Disk Structuring Element
(radius t = 25)

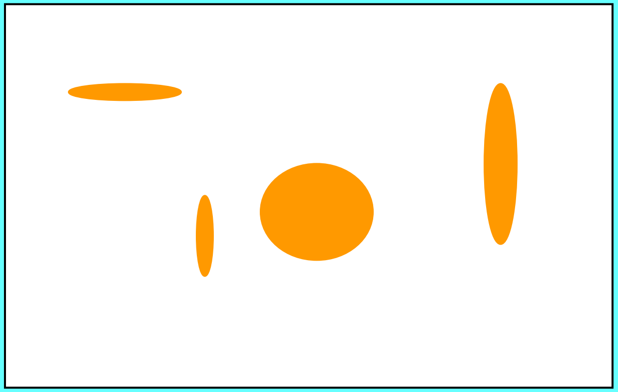
Connected Opening

Eliminate objects for which does not exist *i* and *x* such that $B_i + x \subseteq X$

X



B_1 B_2



$$\bigcup_{i=1}^{2} R(X \circ B_i)$$

Connected opening is a filter:

- idempotent: $\Psi(\Psi(X)) = \Psi(X)$
- anti-extensive: $\Psi(X) \subseteq X$
- increasing: $X \subseteq Y$ implies $\Psi(X) \subseteq \Psi(Y)$
- translation invariant: $\Psi(X+x) = \Psi(X) + x$
- connected

Representation: $$\bigcup_{i=1}^{n} R(X \circ B_i)$$

Connected Granulometries

Model parameterized sieving processes on random sets

A family of operators Ψ_t is a granulometry iff

i- for all $t > 0$, Ψ_t is a connected opening

ii- $r \geq s > 0$ implies $\mathrm{Inv}[\Psi_r] \subseteq \mathrm{Inv}[\Psi_s]$

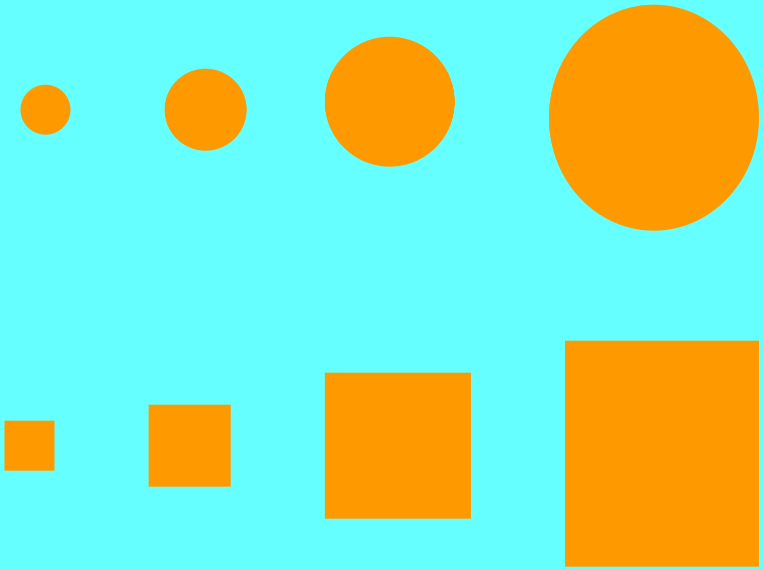
$$Inv[\Psi] = \{X : \Psi(X) = X\}$$

Example

$$\Psi_t(X) = \bigcup_{i=1}^{n} R(X \circ tB_i)$$

tB_1:

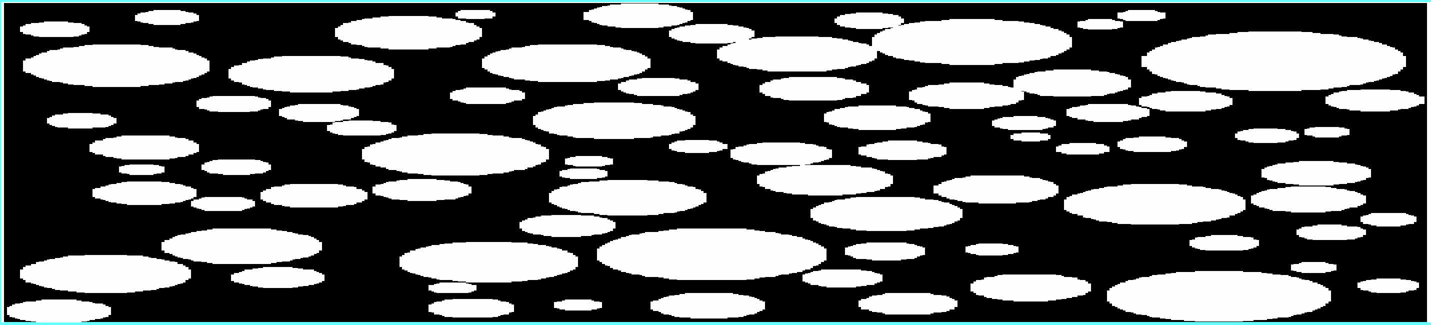
tB_2:



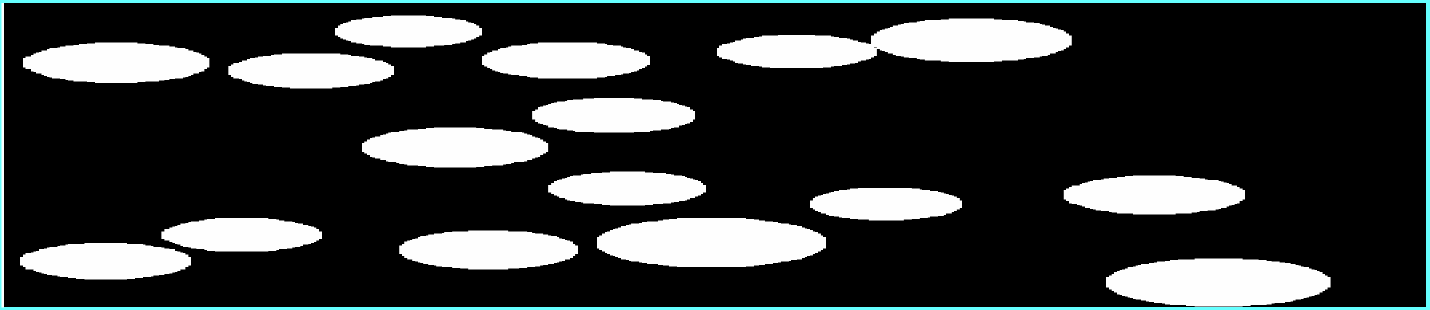
Band Pass Filter

For a connected granulometry $\{\Psi_t\}$, if X is formed as an union of disjoint compact grains and $r < s$, then all grains contained in $\Psi_s(X)$ are also contained in $\Psi_r(X)$, and

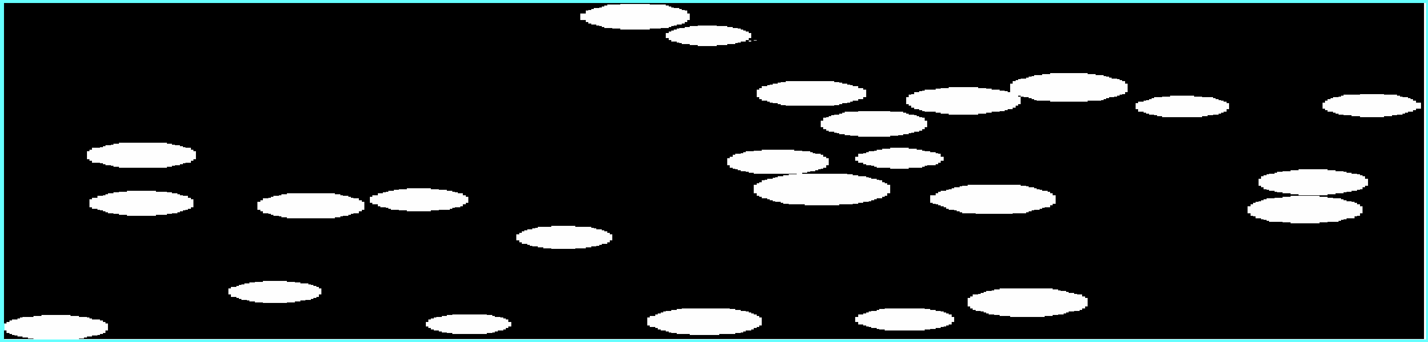
$\Psi_{r,s} = \Psi_r - \Psi_s$ can be viewed as a size band.



Original Image Process



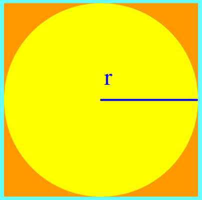
Band pass Filter $\Psi_{25,49} = (\Psi_{25} - \Psi_{45})$



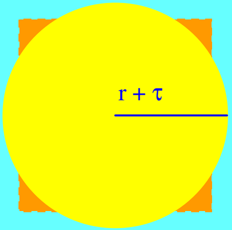
Band pass Filter $\Psi_{15,25} = (\Psi_{15} - \Psi_{25})$

Granulometric Spectrum

Granulometric spectrum of X relative to the granulometry $\{\Psi_t\}$



$$X_t = \bigcap_{\tau>0} [\Psi_t(X) - \Psi_{t+\tau}(X)]$$



The collection $\{X_t\}$ of spectral components forms a partition of X

Granulometric Bandpass Filter

Granulometric band pass filter Ξ corresponding to Π

$$\Xi_{\Pi}(X) = \bigcup_{t \in \Pi} X_t$$

Π is the union of a countable number of intervals

$$\Xi_{\Pi}(X) = \bigcup_{[a,b] \subseteq \Pi} \Psi_{a,b}(X)$$

Granulometric Size Density

For a compact set X, we define the size distribution

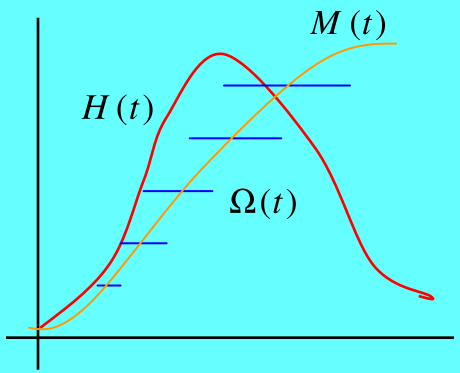
$$\Omega(t) = \nu[X] - \nu[\Psi_t(X)]$$

Mean size distribution (MSD)

$$M(t) = E[\Omega(t)]$$

Granulometric size distribution

$$H(t) = M'(t)$$



Interpretation

Granulometric Filter: spectral shape decomposition; choice of shape bands; reconstruction

Optimal granulometric filters

Image Model

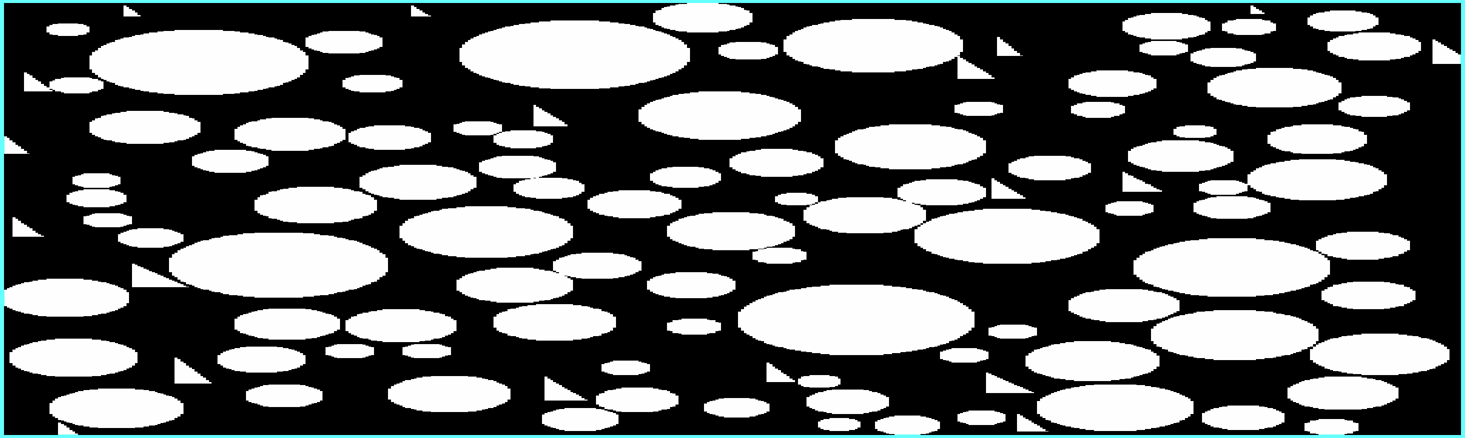
Signal: S; Noise: N; Signal-noise: $S \cup N$, $S \cap N = \varnothing$

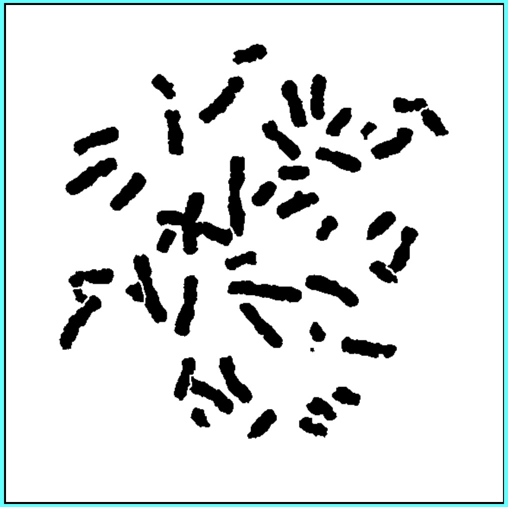
$$S = \bigcup_{i=1}^{I} C[s_i] + x_i \qquad N = \bigcup_{j=1}^{J} D[n_i] + y_j$$

random real numbers: s_i and n_j

random compact grains: $C[s_i]$ and $D[n_j]$

random points: x_i and y_j





Optimization problem

Observation: $S \cup N$, Estimator: $\Xi_\Pi(S \cup N)$

Error: Signal grains erroneously removed and noise grains passed.

Find Π such that Ξ_Π is optimal in $\{\Xi_\Pi\}$, that is, minimizes

$$Er[\Xi_\Pi] = E\left[\nu\left[\Xi_\Pi(S \cup N)\Delta S\right]\right]$$

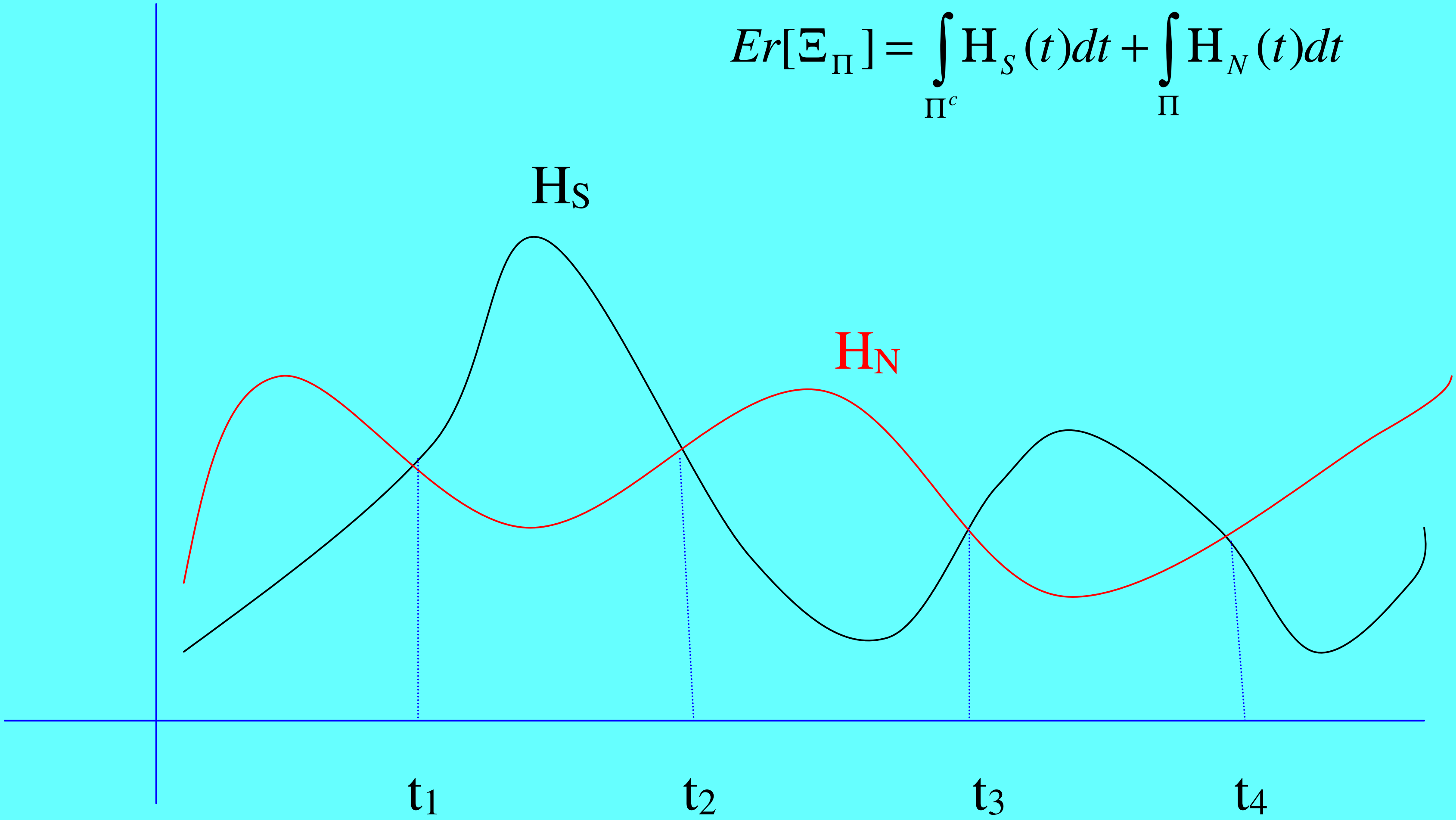
Error Formula

For practical purposes, the MSD is continuously differentiable, and the following theorem holds:

Theorem: The error of the filter Ξ_Π is given by

$$Er[\Xi_\Pi] = \int_{\Pi^c} \mathrm{H}_S(t)dt + \int_{\Pi} \mathrm{H}_N(t)dt$$

$$Er[\Xi_{\Pi}] = \int_{\Pi^c} \mathrm{H}_S(t)dt + \int_{\Pi} \mathrm{H}_N(t)dt$$



$$\pi = [t_1, t_2] \cup [t_3, t_4]$$

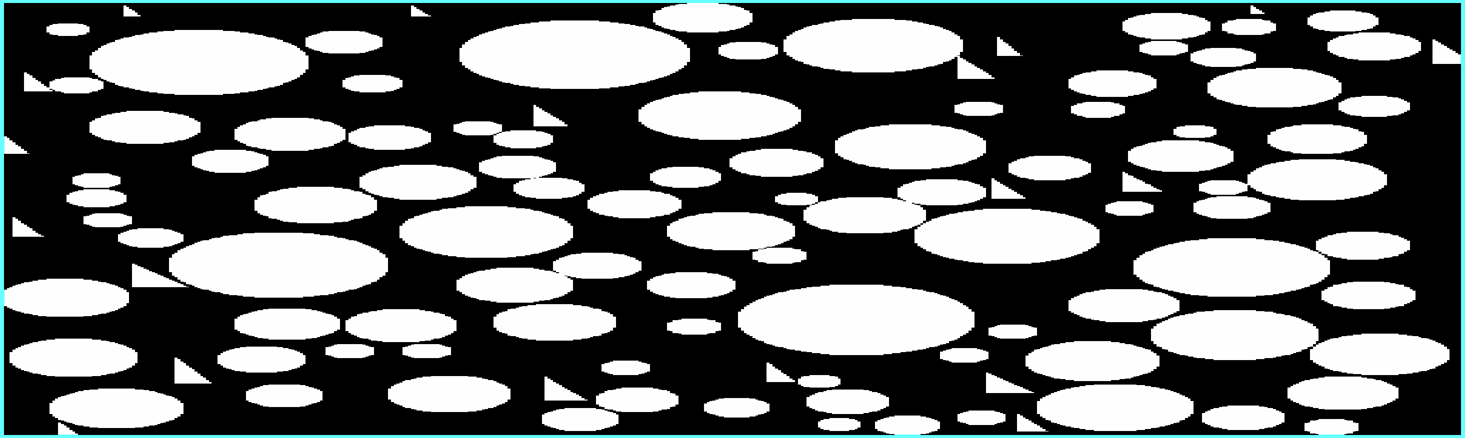
Optimal Filter

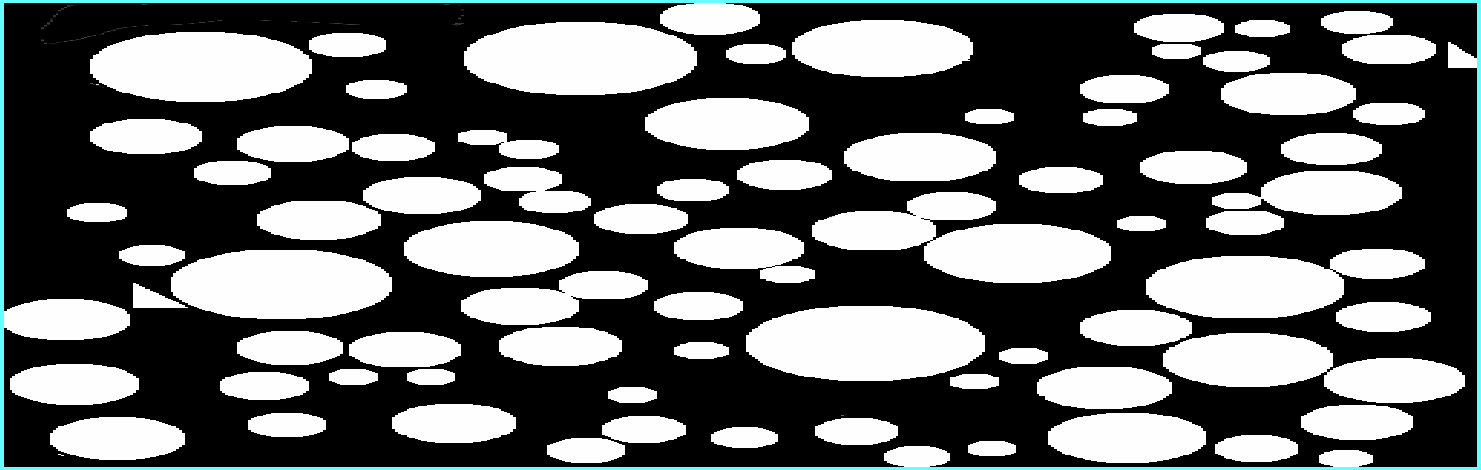
An optimal pass set Π_{opt} is given by:

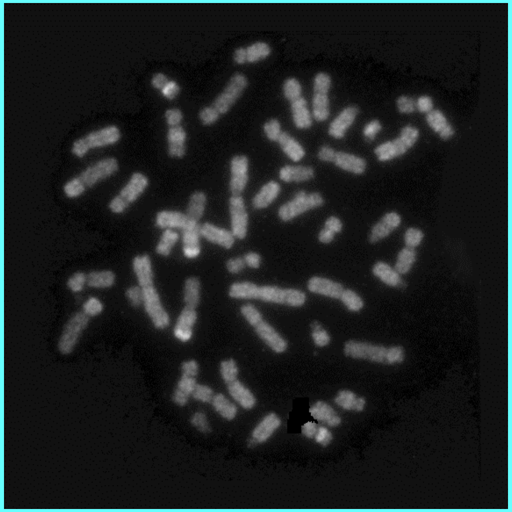
$$\Pi_{opt} = \{ t\colon \mathrm{H}_S(t) \geq \mathrm{H}_N(t) \}$$

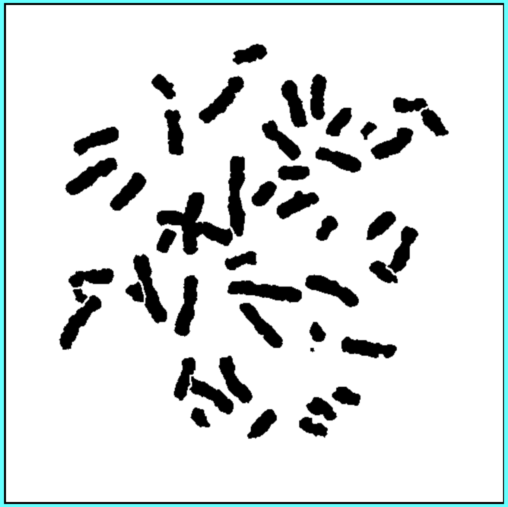
The bandpass form of the optimal filter is:

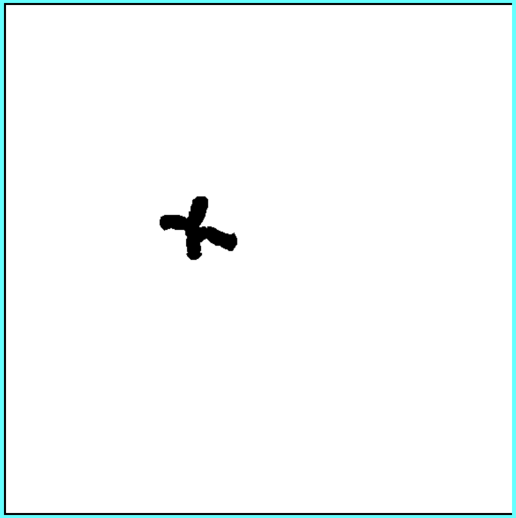
$$\Xi_{opt}(S \cup N) = \bigcup_{[a,b] \subseteq \Pi_{opt}} [\Psi_a(S \cup N) - \Psi_b(S \cup N)]$$

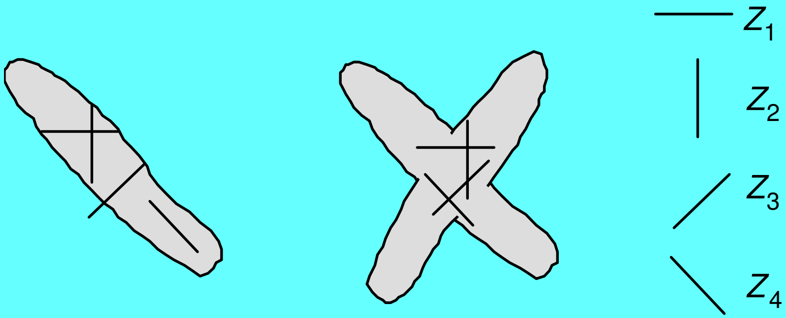











Z_1
Z_2
Z_3
Z_4

Conclusion

- Recalled strategies for designing general non linear filters

- To learn non linear filters it is necessary to estimate the join probability of input and output

- Recalled the Wiener-Holpf formula for designing linear filters

- To learn linear filters it is enough to estimate the power spectral density of the autocorrelation and cross correlation.

- Presented an analogous formula for designing bandpass connected filters

- To learn bandpass connected filters it is enough to estimate the Granulometric Size Distribution of Signal and Noise

- The result holds by analogous facts observed in the increasing case: signal decomposition and operator representation

- There is no known generalization of these results for

$$\Psi_{\bar{t}}(X) = \bigcup_{i=1}^{n} R(X \circ t_i B_i)$$

where t_i is a projection of the vector $\bar{t}$

- The result is generalized for

$$\Psi_{\bar{t}}(X) = \bigcap_{i=1}^{n} R(X \circ t_i B_i)$$