From the Sup-Decomposition to a Sequential Decomposition

Ronaldo Fumio Hashimoto

Junior Barrera

Edward R. Dougherty

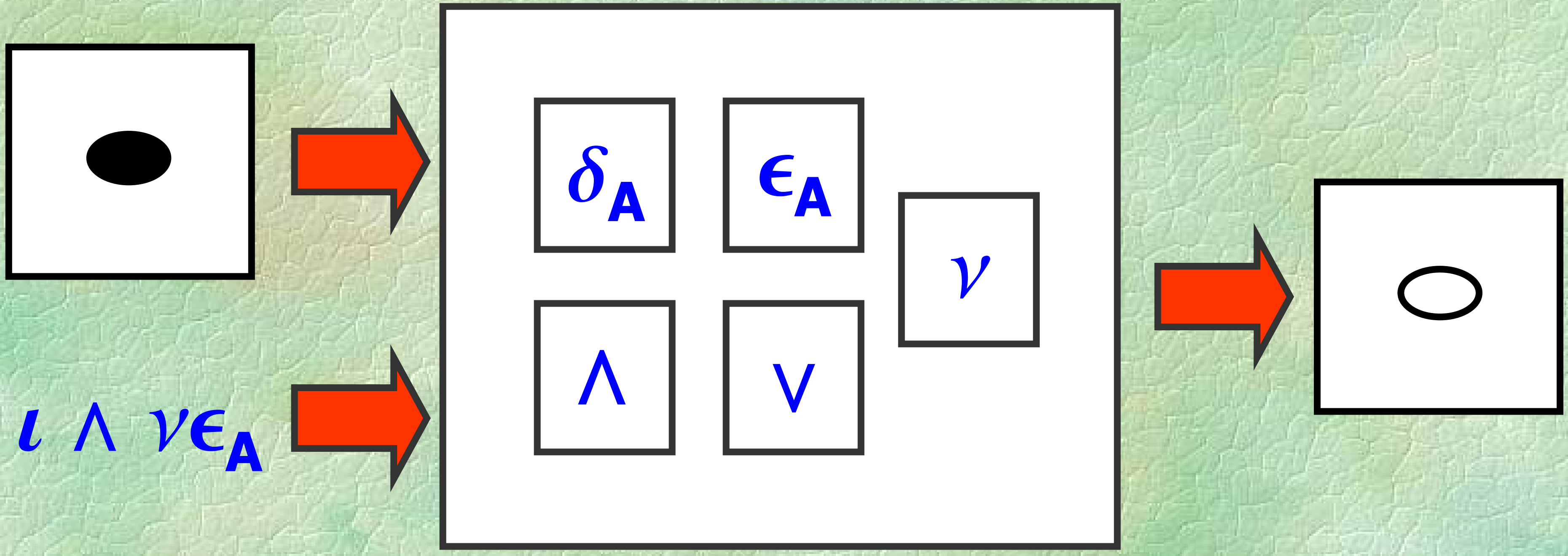
Computer Science Department
University of São Paulo - Brazil

Outline

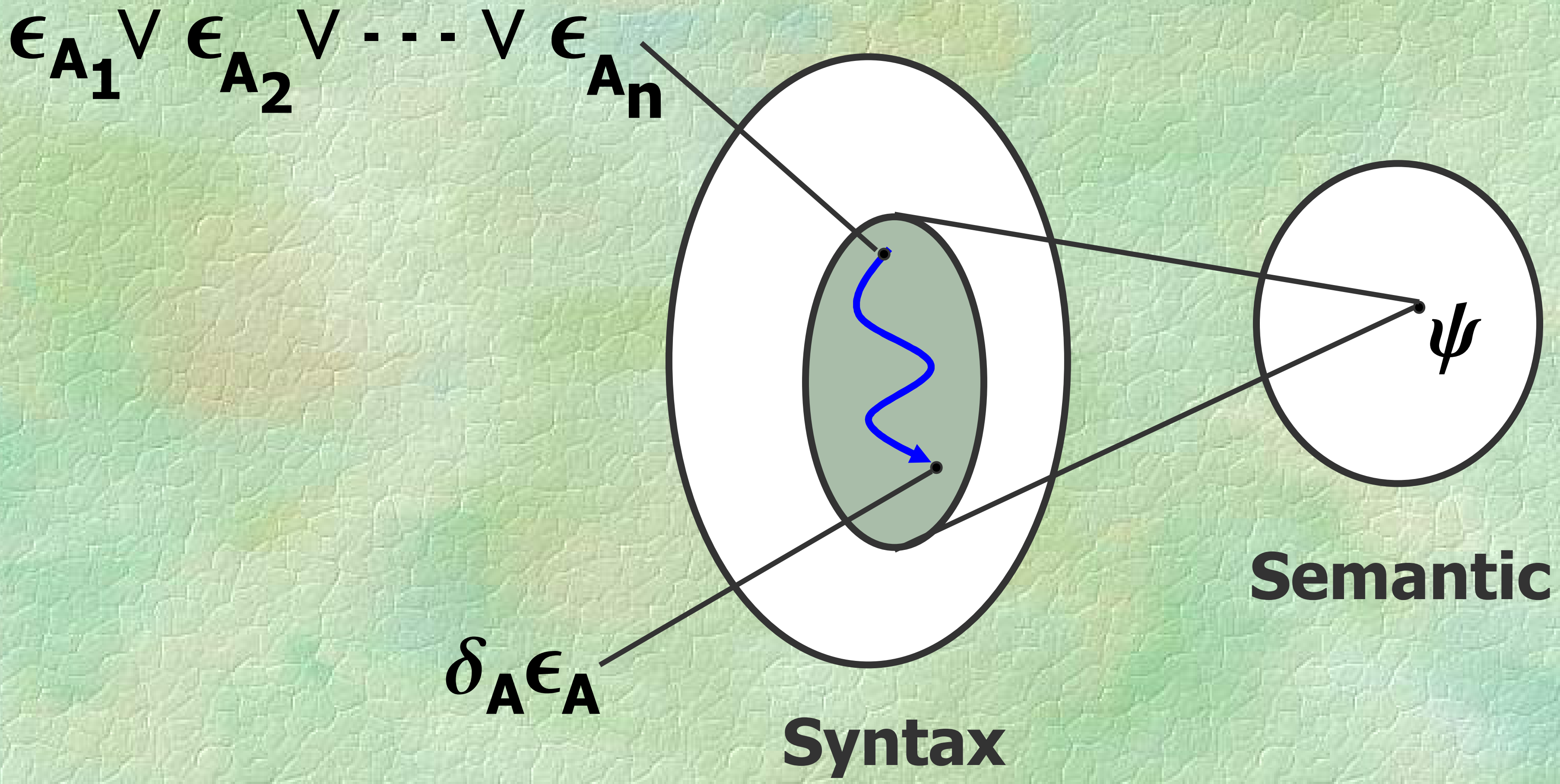
- Introduction
- Collection of Maximal Intervals
- Minkowski Factorization Equation
- Bounds
- Fixed Right Extremity Simplification
- W-Operators
- Sup-Decomposition to Sequential Decomposition
- Conclusion

Introduction

Morphological Machine



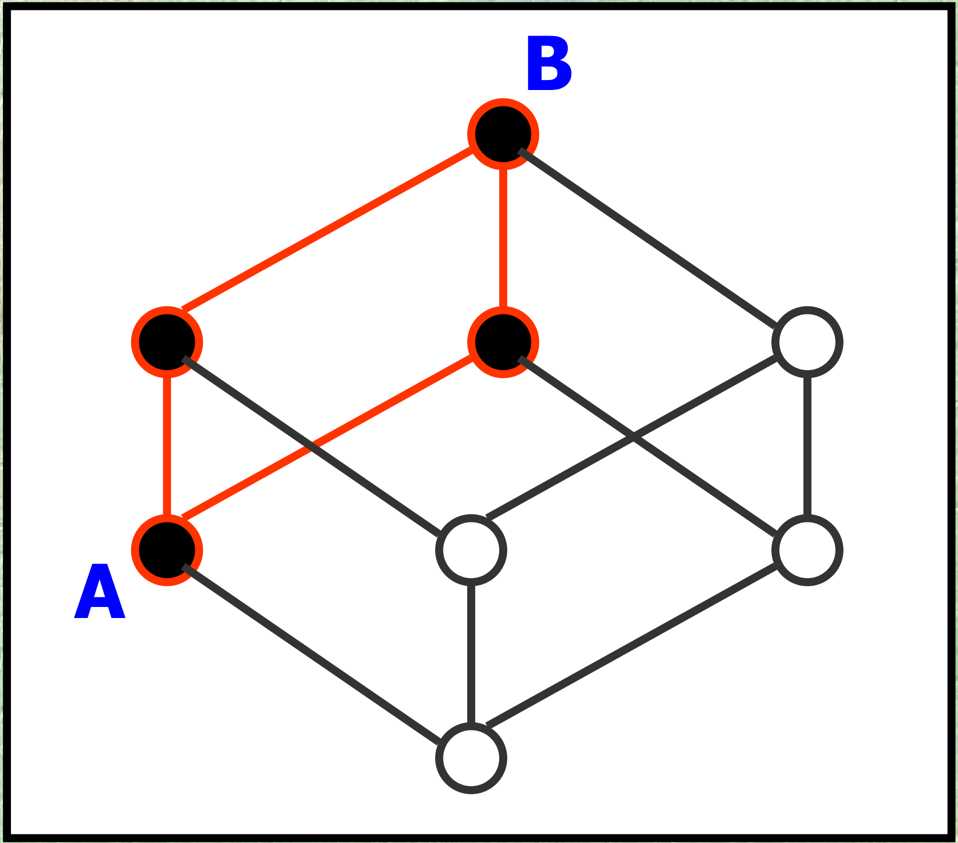
Introduction



Interval

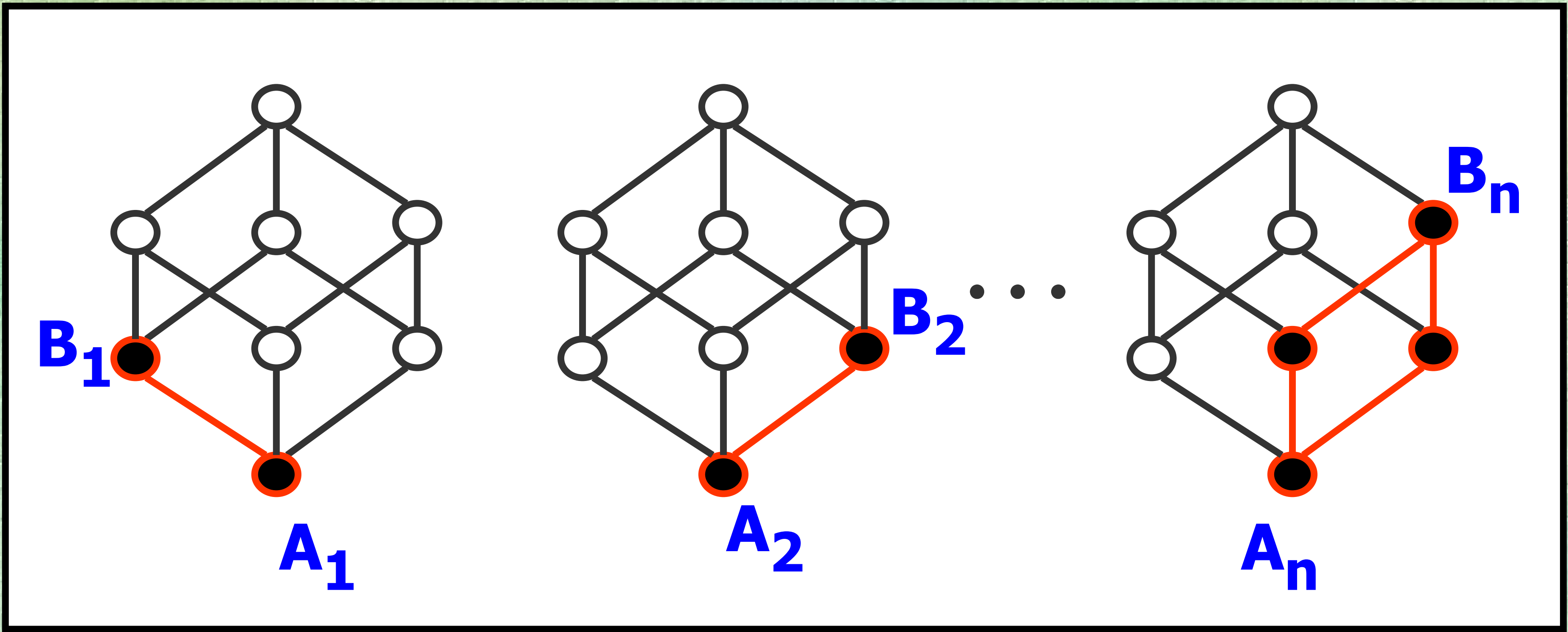
$$A, B \in P(W) : A \subseteq B$$

$$[A,B]=\{X \in P(W) : A \subseteq X \subseteq B\}$$



Collection of Intervals

$$X_W = \{[A_1,B_1], [A_2,B_2], \ldots, [A_n,B_n]\}$$

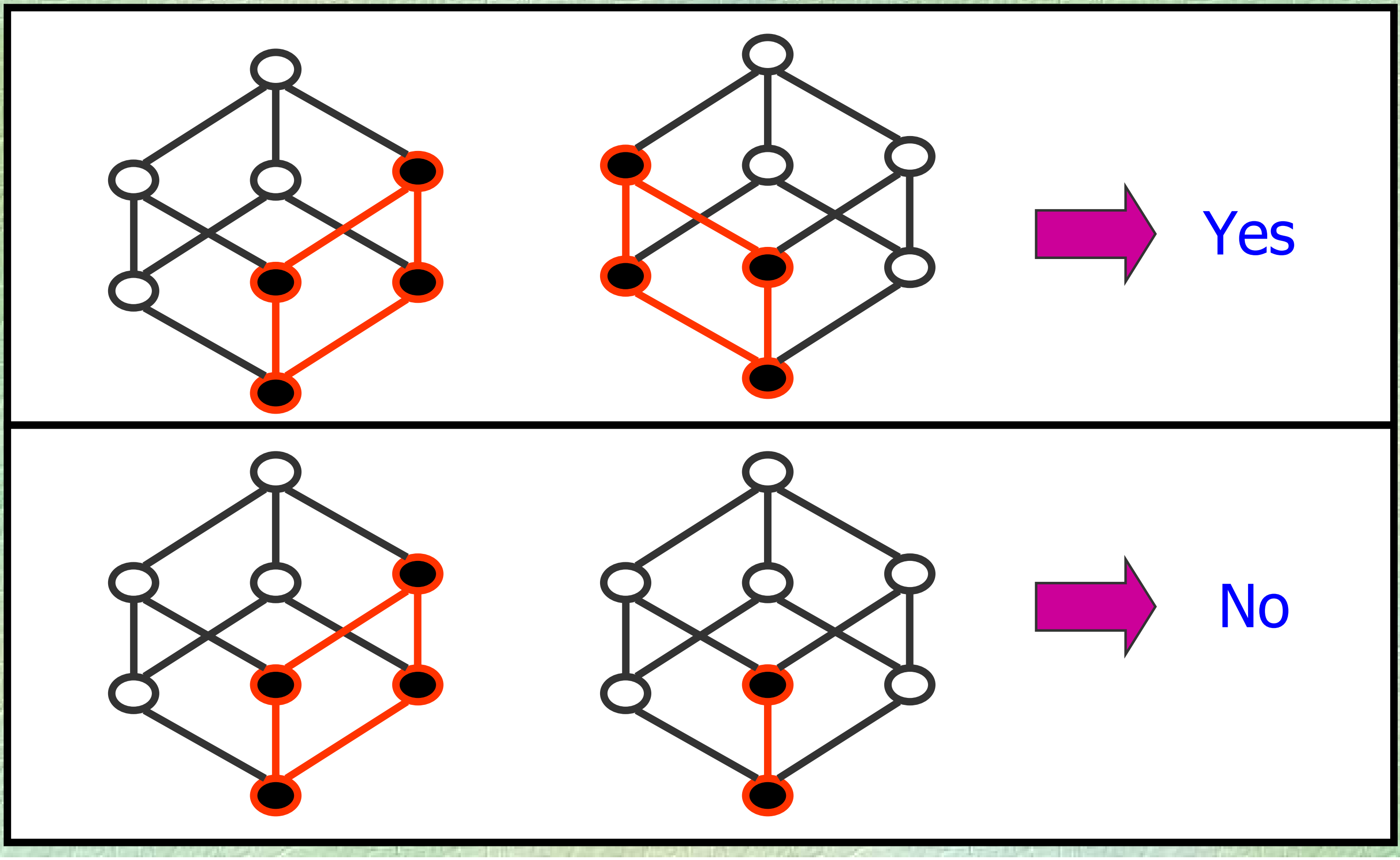


Collection of Maximal Intervals

X_W is a Coll. of Maximal Intervals iff

$\forall [A,B] \in X_W$, $\nexists [A',B'] \in X_W$:

$[A',B'] \neq [A,B]$ and $[A,B] \subseteq [A',B']$.



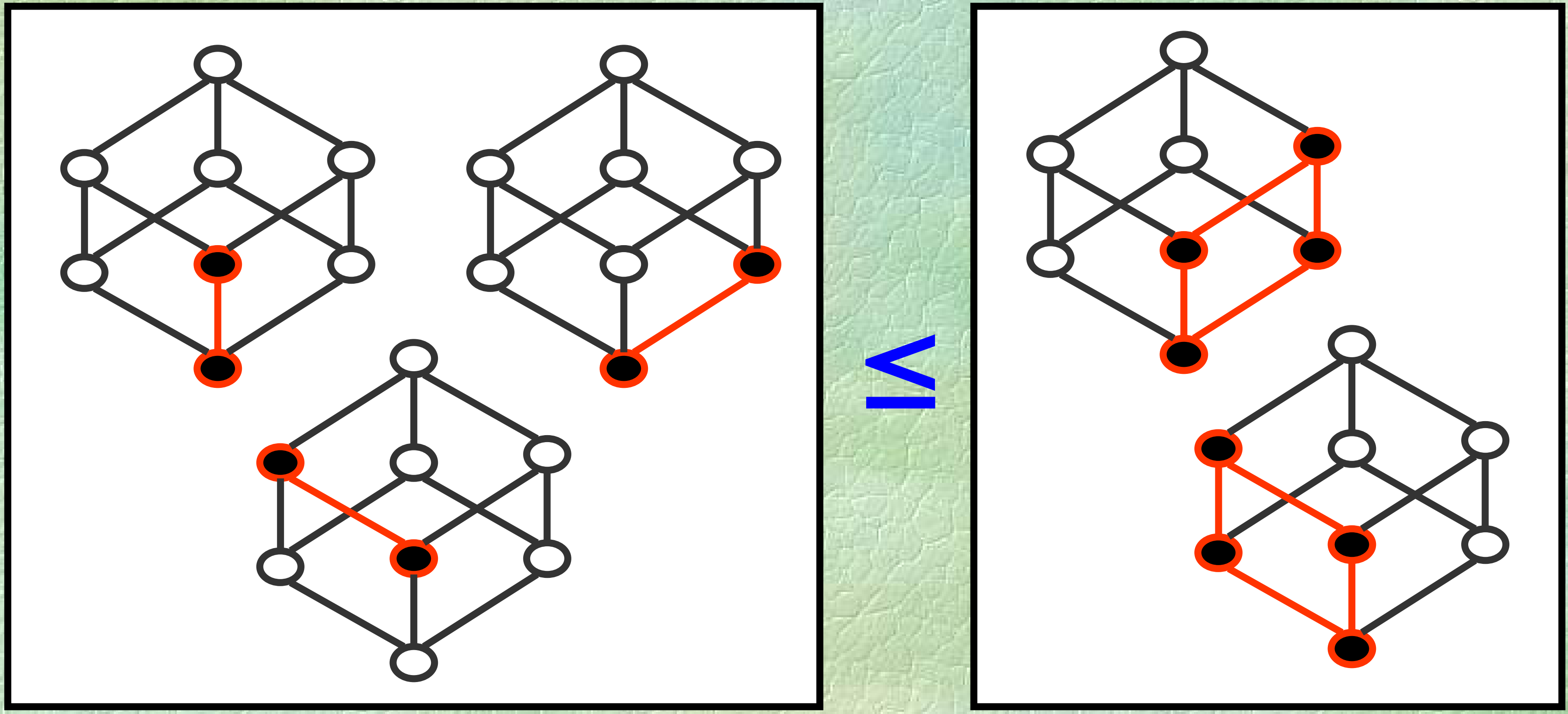
Collection of Maximal Intervals

Let Π_W denote the set of all collections of maximal intervals contained in P(P(W)).

Collection of Maximal Intervals

Partial Order on the elements in Π_W

$$X \leq Y \Leftrightarrow \forall [A,B] \in X, \exists [A',B'] \in Y: [A,B] \subseteq [A',B']$$



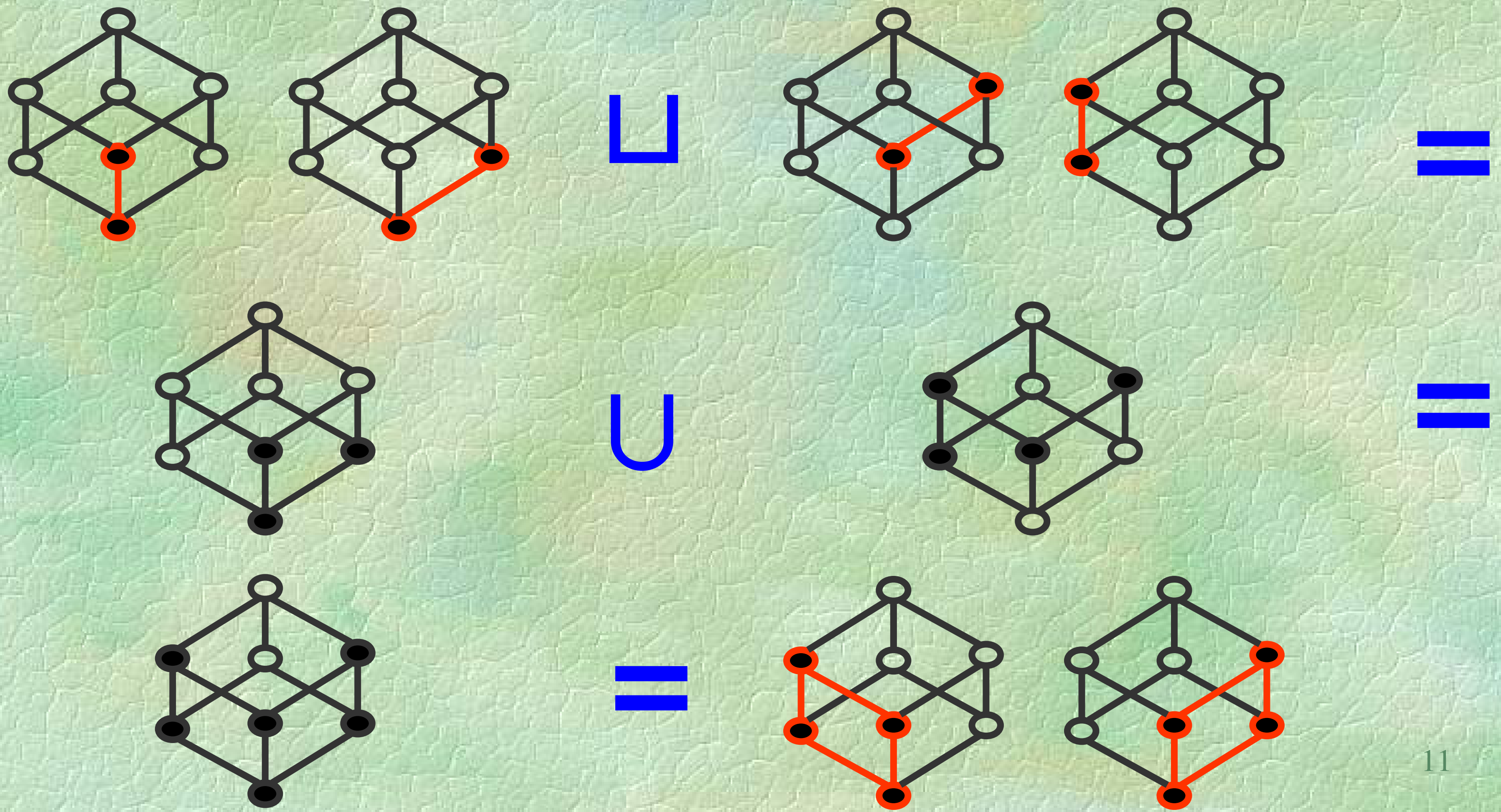
Collection of Maximal Intervals

$(\Pi_W, \leq)$ is a Complete Boolean Lattice

Collection of Maximal Intervals

Supremum operation: $\forall X, Y \in \Pi_W$

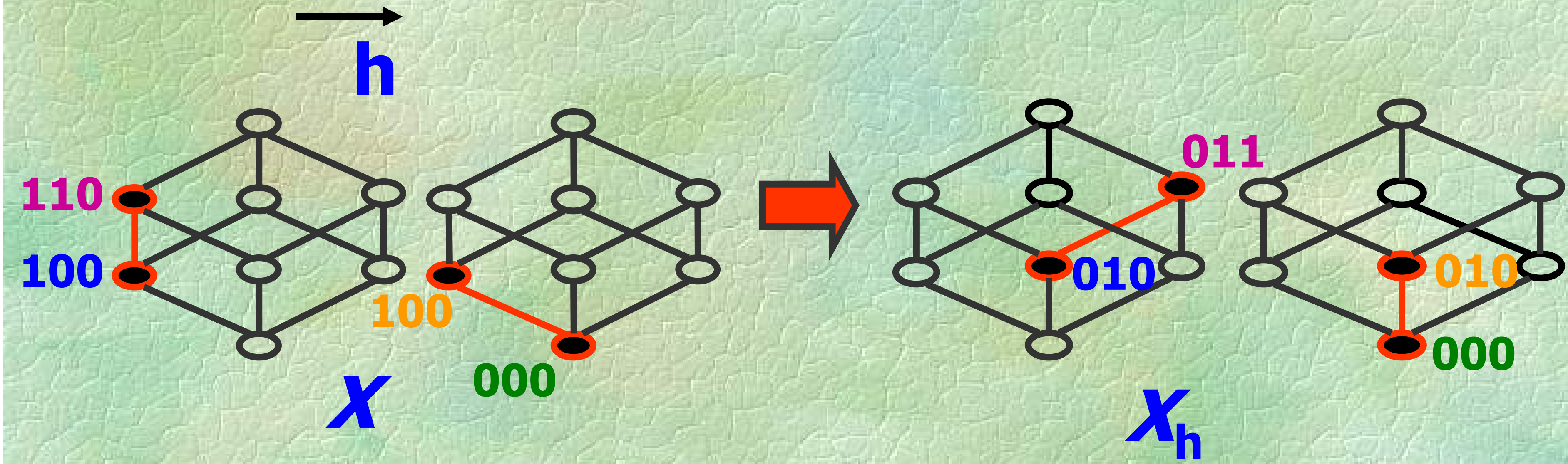
$$X \sqcup Y = M(\cup(X) \cup \cup(Y))$$



Collection of Maximal Intervals

Some operations on $X_W \in \Pi_W$:

$$(X_h)_{W+h} = \{ [A_h, B_h] : [A, B] \in X \}$$



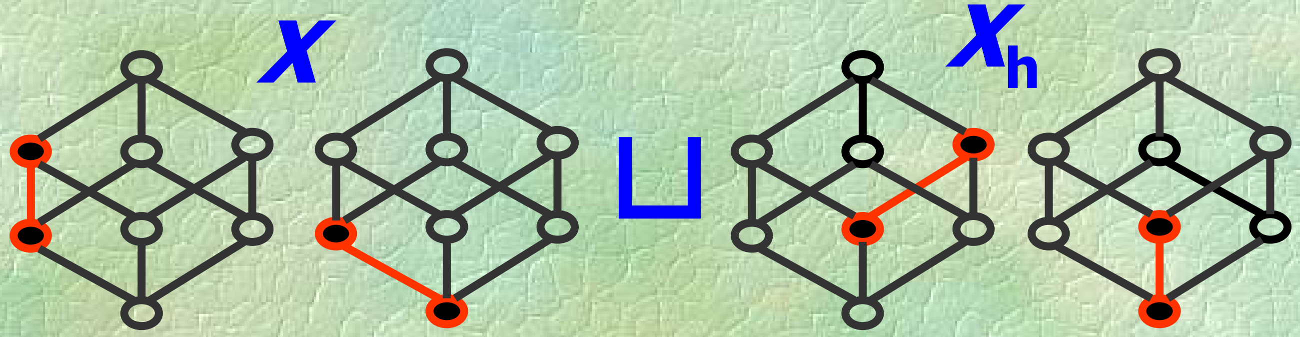
Collection of Maximal Intervals

Some operations on $X_w \in \Pi_w$:

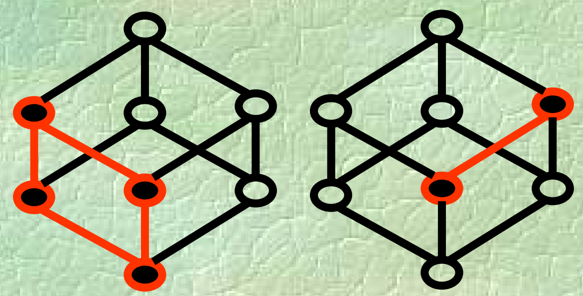
$$X_w \oplus C = \sqcup\{\ (X_h)_{w\oplus C} : h \in C\ \}$$

$\vec{h}$ $C=\{o,h\}$

$X \oplus C = X \sqcup X_h =$



$=$



Minkowski Factorization Equation

Given $Y_{W'} \in \Pi_{W'}$ and $C \in P(E)$,
find $W \in P(E)$ and $X_W \in \Pi_W$ such that

$$X_W \oplus C^t = Y_{W'} \qquad (1)$$

$$W = W' \ominus C^t$$

$$L_W \leq X_W \leq U_W$$

Lower bound

Upper bound

Algorithm for Finding X_W

$C \in P(E)$ $Y_{W'} \in \Pi_{W'}$ $W = W' \ominus C^t$

Find $X_W \in \Pi_{W'}$ such that $X_W \oplus C^t = Y_{W'}$.

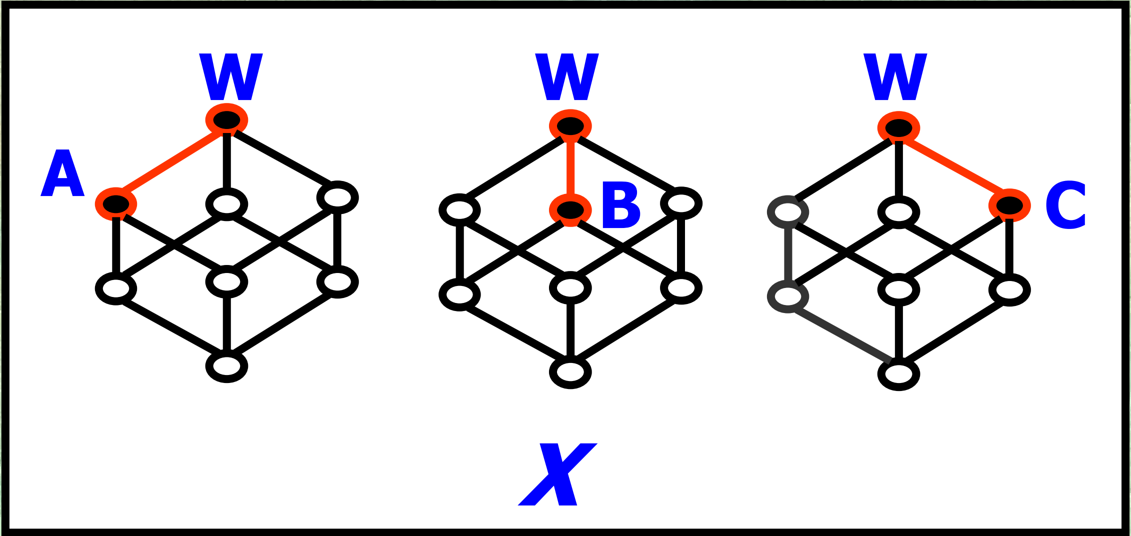
Algorithm Search 1:

Compute U_W
For each $L_W \in \Theta_C(Y_{W'})$ do
  For each X_W such that $L_W \leq X_W \leq U_W$
    If $X_W \oplus C^t = Y_{W'}$
      Output X_W

Fixed Right Extremity

$$\mathfrak{R}_W \subseteq \Pi_W$$

$X \in \mathfrak{R}_W \Leftrightarrow$ **The right extremity of any interval in *X* is the window W.**



$X = \{[A], [B], [C]\} \in \mathfrak{R}_W$

Lower Bound Simplification

Property:

$$\forall X_W \in \Re_W \text{ such that } X_W \oplus C^t = Y_{W'}$$

$$\exists L_W \in \Phi_C(Y_{W'}),\ L_W \le X_W \text{ and } L_W \oplus C^t = Y_{W'}$$

Algorithm for Finding X_W

$C \in P(E)$ $\quad Y_{W'} \in \mathcal{R}_{W'}$ $\quad W = W' \ominus C^t$

Find $X_W \in \mathcal{R}_{W'}$ such that $X_W \oplus C^t = Y_{W'}$.

Algorithm Search 2:

Compute U_W
For each $L_W \in \Phi_C(Y_{W'})$ do
  If $L_W \oplus C^t = Y_{W'}$
    For each X_W such that $L_W \leq X_W \leq U_W$
      If $X_W \oplus C^t = Y_{W'}$
        Output X_W

Feasible Sets

$C \in P(E)$ **is feasible**

$\Updownarrow$

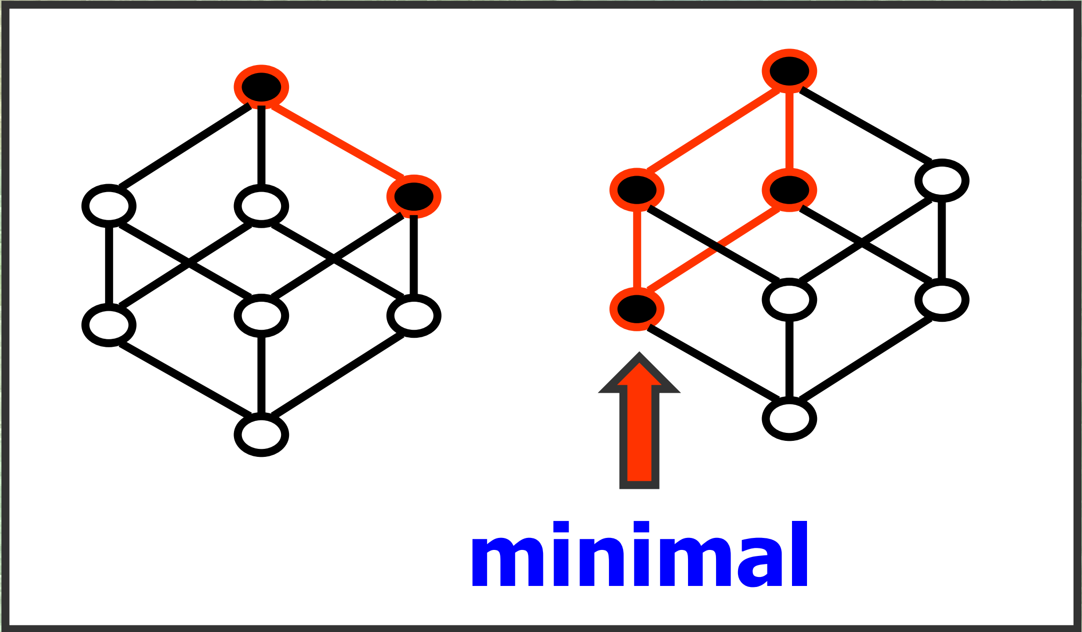
$X_W \oplus C^t = Y_{W'}$

has at least one solution.

Minimal Left Extremity

$X_W \in \mathcal{R}_W$ $\quad\quad$ $[A] \in X_W$

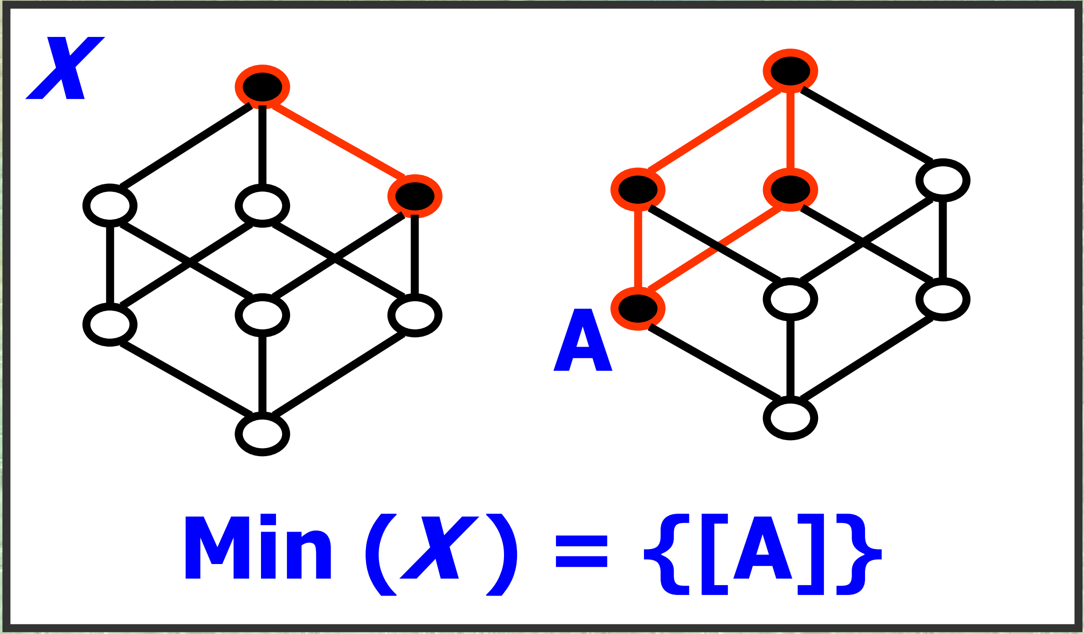
[A] is minimal in $X \Leftrightarrow \forall [B] \in X_W$, $|A| \leq |B|$



Minimal Left Extremity

$X \in \mathcal{R}_W$ $[A] \in X$

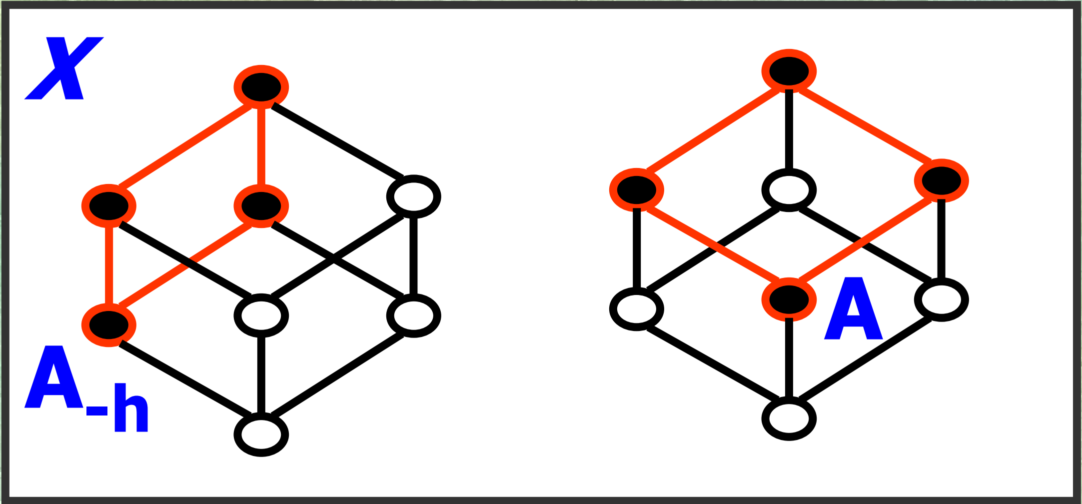
Min (X) = {[A] $\in X$: [A] is minimal in X}



Minimal Left Extremity

$\mathcal{X} \in \mathcal{R}_W$ $\qquad$ $A \in P(E)$

$$S_A(\mathcal{X}) = \{h \in E : [A_{-h}] \in \mathcal{X}\}$$



$\vec{h}$

$S_A(\mathcal{X}) = \{o,h\}$

Invariant

$A, B \in P(E)$

B is an invariant of A iff
$B = (A \ominus B) \oplus B$

Algorithm for Finding C and X_W

Given $Y_{W'} \in \mathfrak{R}_{W'}$, find $C \in P(E)$ and

$X_W \in \mathfrak{R}_{W'}$ such that $X_W \oplus C^t = Y_{W'}$.

Algorithm Search_All:

Let $[A] \in \mathrm{Min}\,(Y)$:

$\forall [B] \in \mathrm{Min}\,(Y),\ |S_A(Y)| \le |S_B(Y)|$

For each $C \subseteq S_A(Y)$ do

If C is an invariant of $S_A(Y)$

Let $\{X_1, \cdots, X_n\}$ be

the output of Search_2 (Y, C)

For i=1 to n output the pair (C, X_i)

Operator

An operator is a mapping from P(E) to P(E)

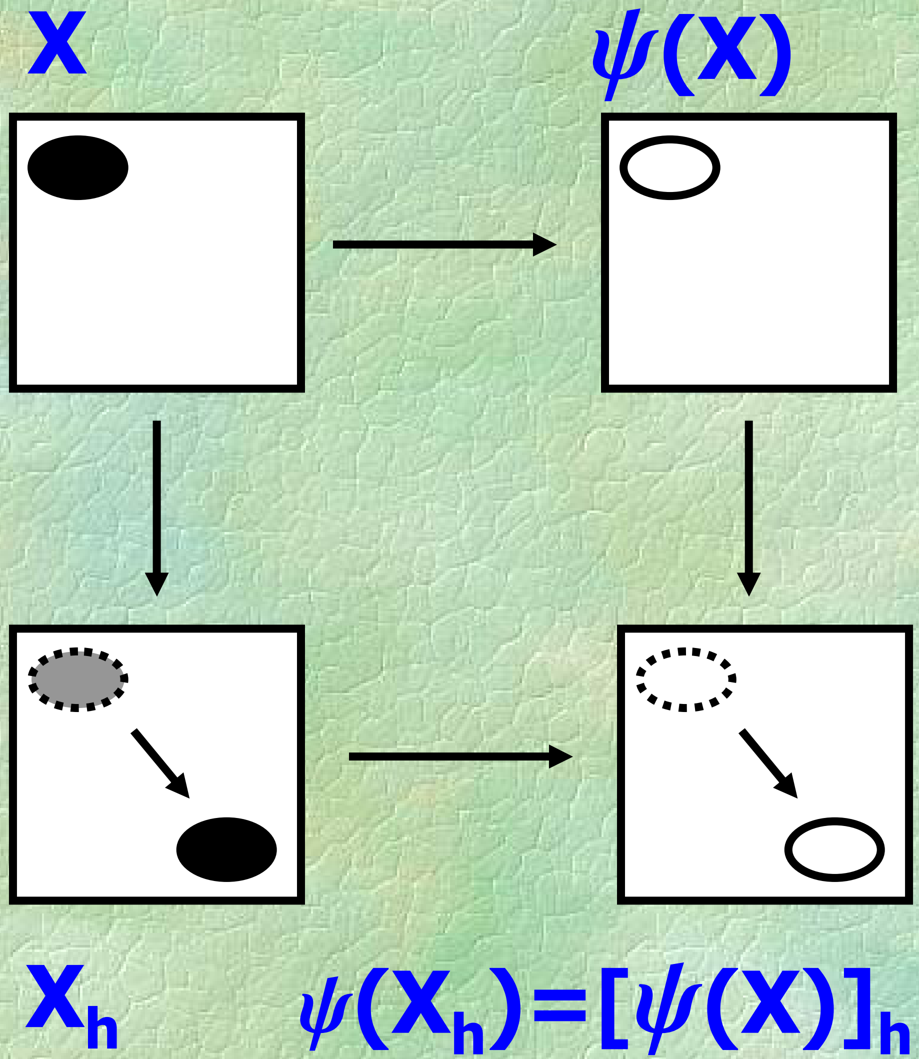
$$\psi : P(E) \rightarrow P(E)$$

Translation Invariance

- **Translation of X by h:**

  $X_h = \{x+h : x \in X\}$

- ψ is translation invariant iff $\psi(X_h)=[\psi(X)]_h$

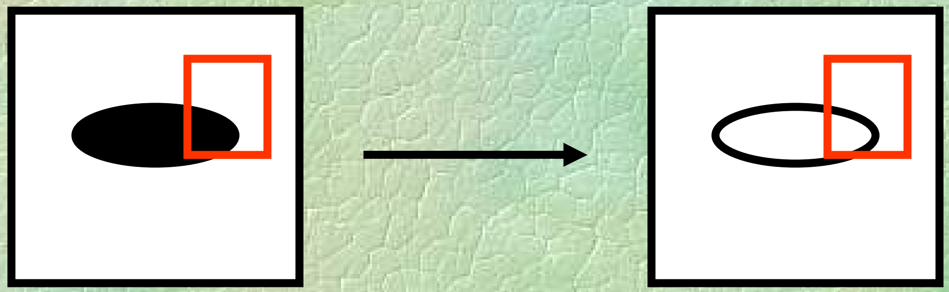


Local Definition

Window : $W \subseteq E$

➡ **An operator is locally defined within W iff**

$$h \in \psi(X) \Leftrightarrow h \in \psi(X \cap W_h)$$



$X \cap W_h$ $\qquad\qquad$ $\psi(X \cap W_h)$

W-Operators

Translation invariance

+

local definition within W

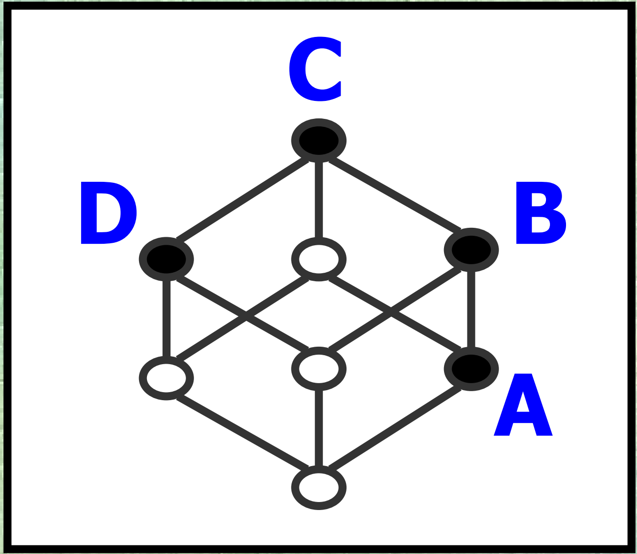
=

W-operators

Kernel

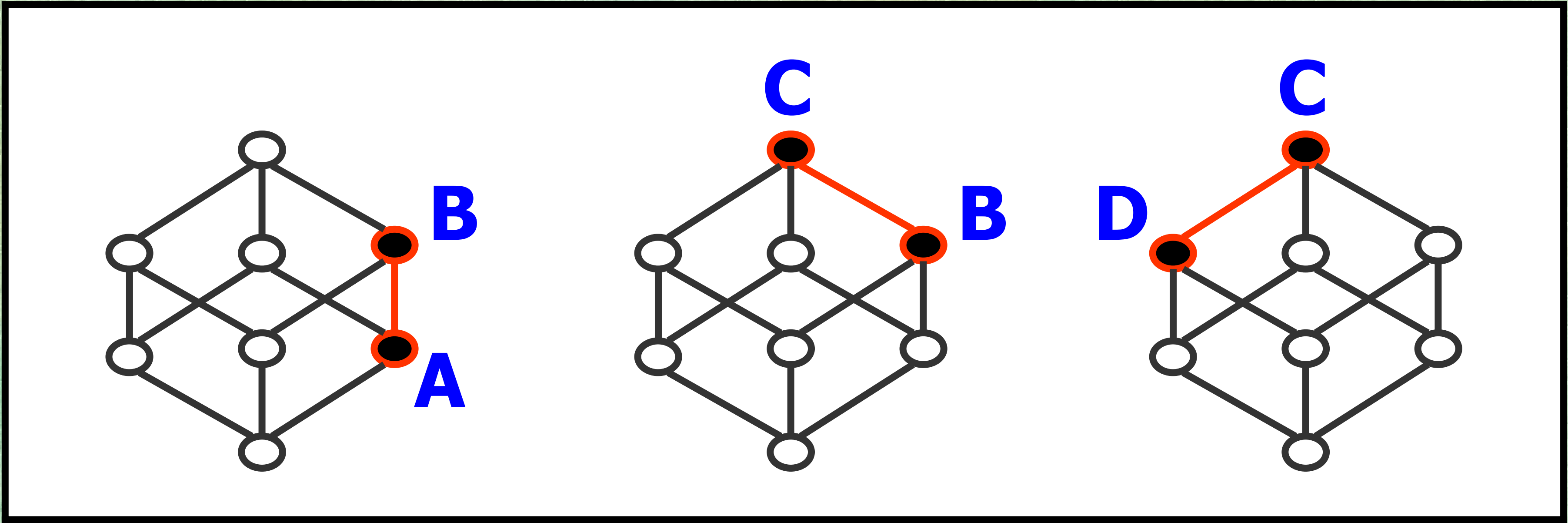
$$\mathcal{K}(\psi) = \{X \in P(W) : o \in \psi(X)\}$$

$$\mathcal{K}(\psi) = \{A, B, C, D\}$$



Basis

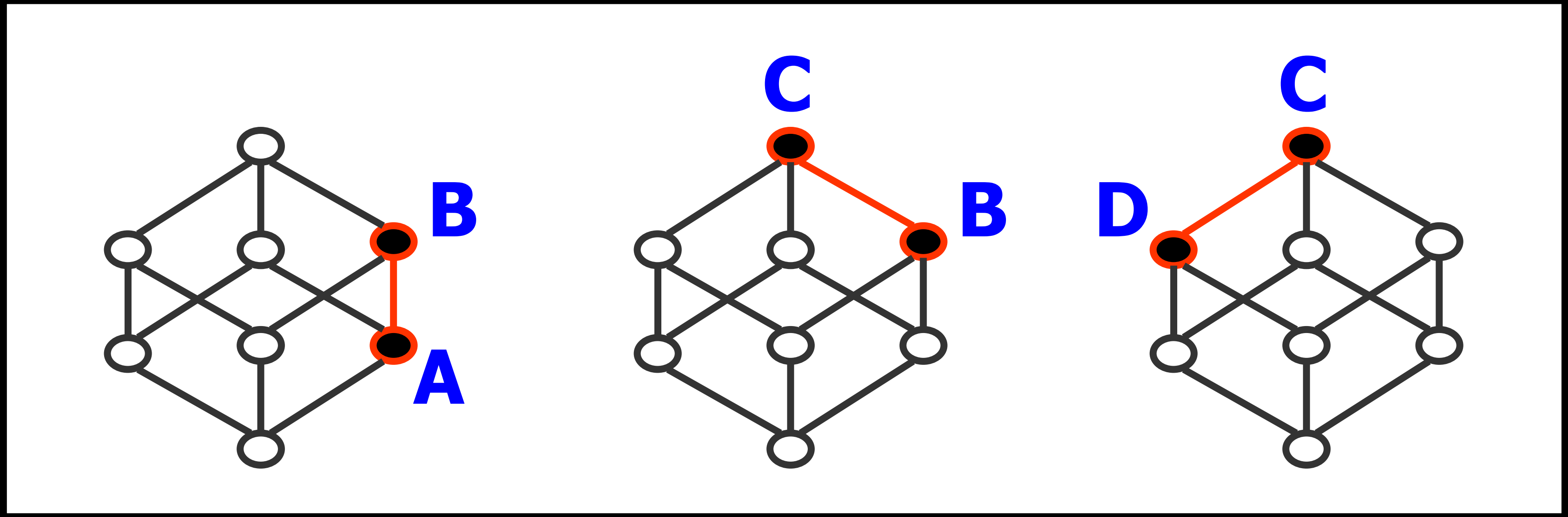
$$B(\psi) = M(\mathcal{K}(\psi))$$



$$B(\psi) = \{[A,B], [B,C], [D,C]\}$$

Sup-Decomposition

$$\psi = \vee\{\lambda_{A,B} : [A,B] \in B(\psi)\}$$

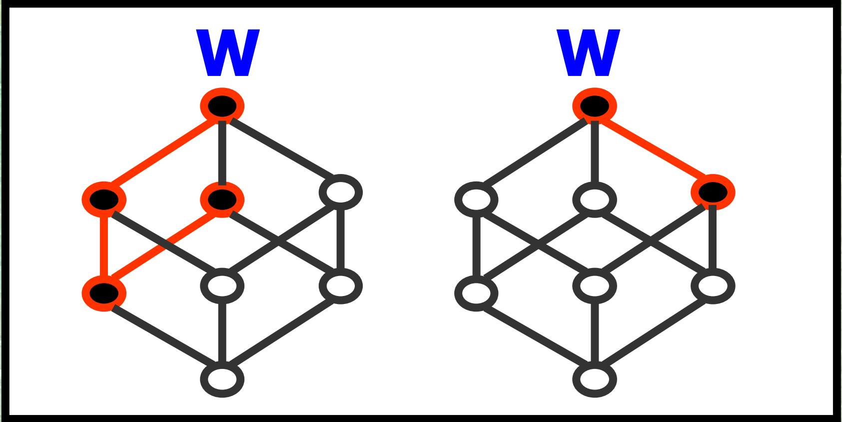


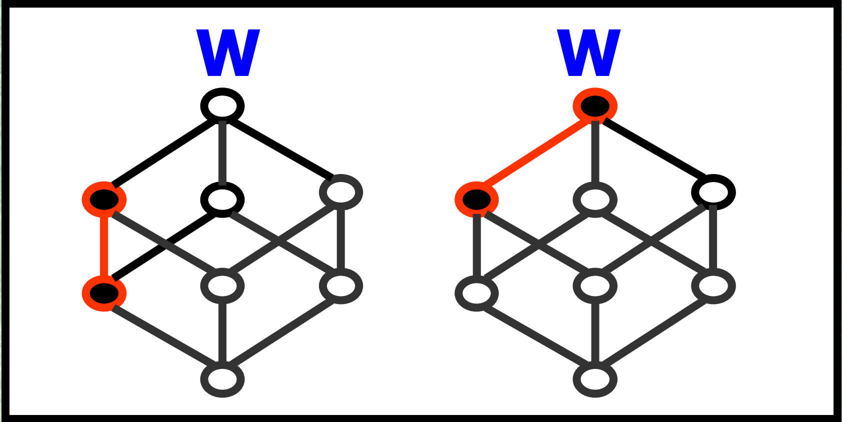
$$\psi = \lambda_{A,B} \vee \lambda_{B,C} \vee \lambda_{D,C}$$

Increasing Operator

ψ is increasing $\Leftrightarrow$ B(ψ) $\in \mathcal{R}_W$

Collection of Maximal Fixed Right Extremity Intervals

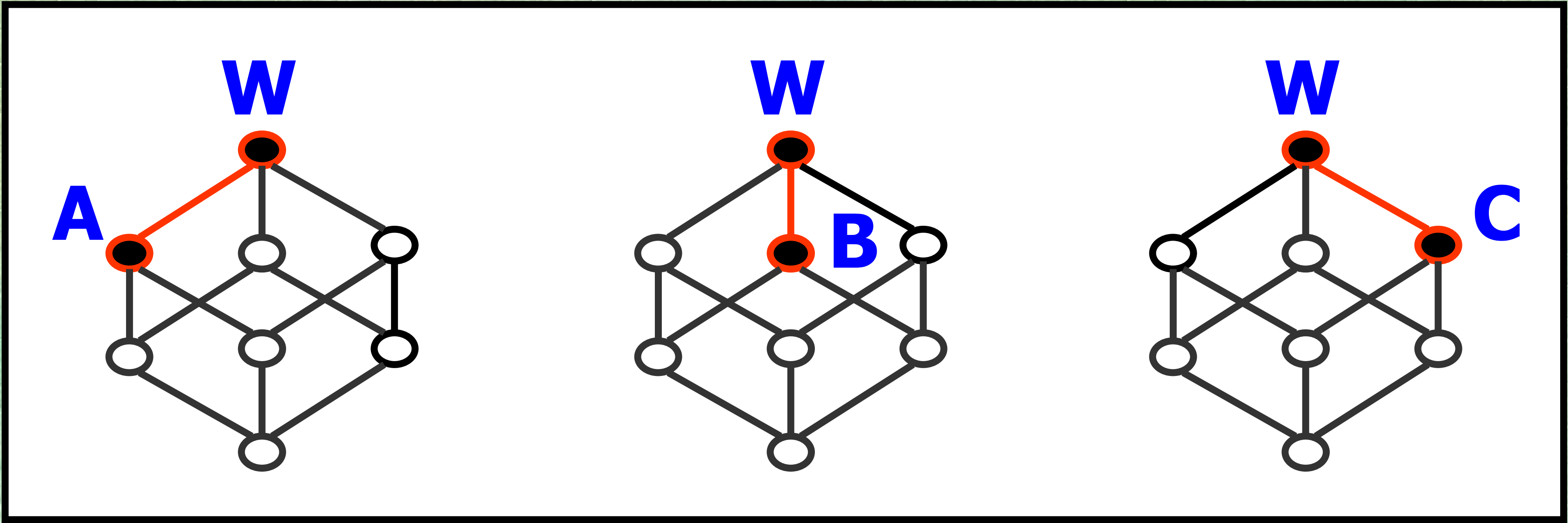




Increasing **Non-increasing**

Increasing Operator

$$\psi = \vee\{\epsilon_A : [A,W] \in B(\psi)\}$$



$$\psi = \epsilon_A \vee \epsilon_B \vee \epsilon_C$$

Dual Operator

ν (X) = X^c ⇦ **Negation Operator**

$\psi^* = \nu \, \psi \, \nu$ ⇦ **Dual Operator**

Compositions of Erosions and Dilations

$$\psi \in \Upsilon_{W}$$

$$\Updownarrow$$

ψ can be built by compositions of erosions and dilations

$$\psi = \epsilon_{A}\delta_{B}\epsilon_{B}\cdots\epsilon_{A}\delta_{B}\delta_{A}\epsilon_{C}\delta_{C}$$

Compositions of Erosions and Dilations

Properties

$\psi \in \Upsilon_{\mathbf{W}} \Rightarrow \psi$ **is increasing**

$\psi \in \Upsilon_{\mathbf{W}} \Leftrightarrow \psi^* \in \Upsilon_{\mathbf{W}}$

Transformation

$$\psi = \delta_C \varphi \Leftrightarrow B(\psi) = B(\varphi) \oplus C^t$$

$$\psi = \epsilon_{C^t} \varphi \Leftrightarrow B(\psi^*) = B(\varphi^*) \oplus C^t$$

Transformation

$$\psi \in \Upsilon_W$$

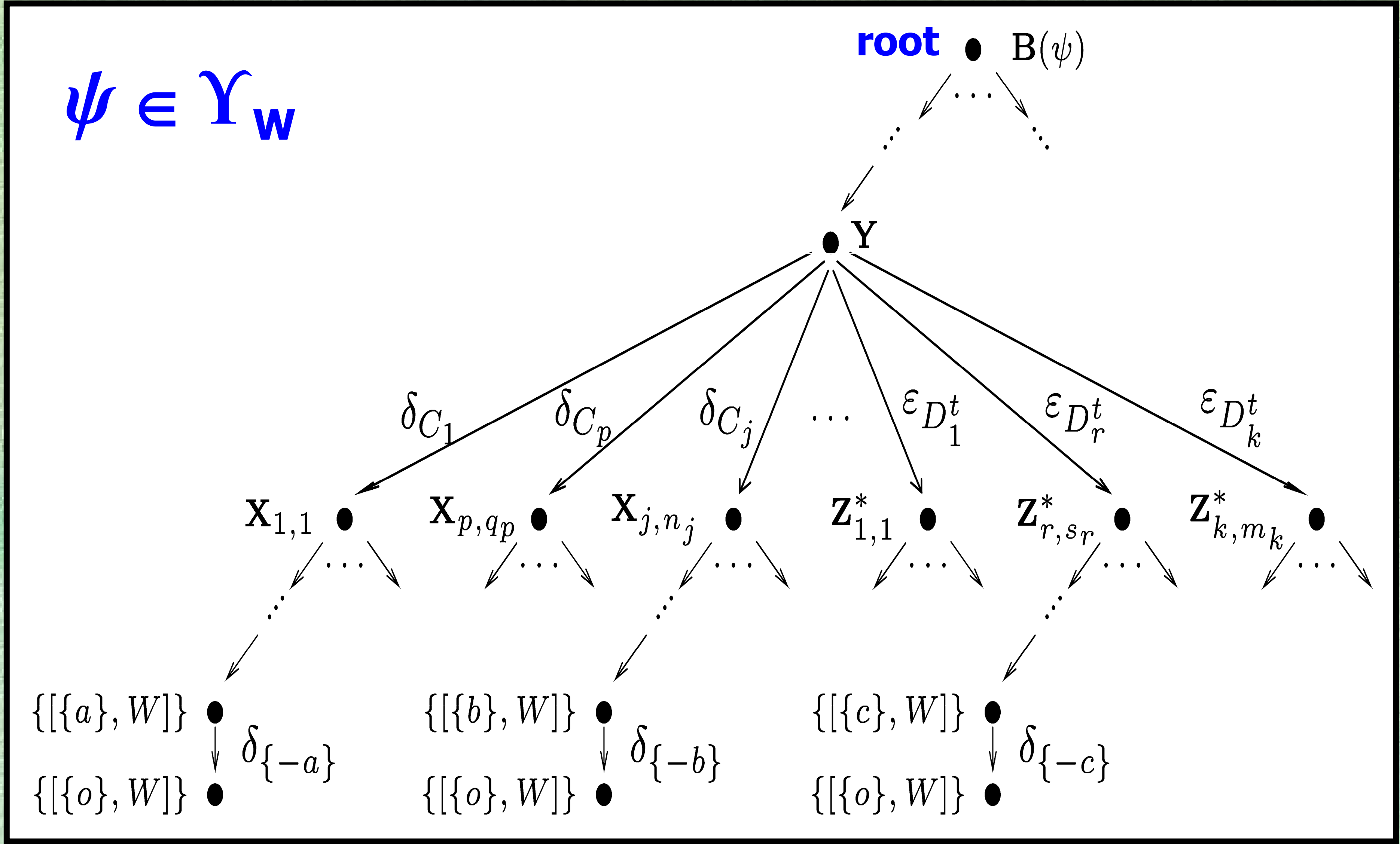
If (C, X) is an Output of Search_All (B(ψ))

$$\Rightarrow \psi = \delta_C \varphi , \varphi \in \Upsilon_W , B(\varphi) = X$$

If (D, X) is an Output of Search_All (B(ψ*))

$$\Rightarrow \psi = \epsilon_D t \varphi , \varphi \in \Upsilon_W , B(\varphi^*) = X$$

Representation Tree



Example of Application

$\psi \in \Upsilon_W$

$Y = B(\psi)$

Representation Tree

$\bullet\ Y_1 = Y$

$W = W_1 = 11\underline{1}11$

$\mathbf{Y}_1 = \mathbf{Y} = \{[00\underline{1}01], [10\underline{1}00], [01\underline{0}00], [00\underline{0}10]\}$

P Q R S

$\psi = \epsilon_P \vee \epsilon_Q \vee \epsilon_R \vee \epsilon_S$

(a)

Example of Application

Representation Tree

$$\bullet\ Y_1 = Y$$

$$\delta_{C_1} \downarrow$$

$$\bullet\ Y_2 = X_1$$

$W = W_1 = 11\underline{1}11$

$\mathbf{Y}_1 = \mathbf{Y} = \{[00\underline{1}01], [10\underline{1}00], [01\underline{0}00], [00\underline{0}10]\}$

$\mathbf{Y}_1^* = \mathbf{Y}^* = \{[11\underline{0}11], [01\underline{1}10]\}$

Output of SEARCH_ALL $(\mathbf{Y}_1)$:

$C_1 = 10\underline{1}$

$\mathbf{Y}_2 = \mathbf{X}_1 = \{[10\underline{1}], [01\underline{0}]\}$

$W_2 = W \ominus C_1^t = 11\underline{1}$

Output of SEARCH_ALL $(\mathbf{Y}_1^*)$:

$\emptyset$.

(b)

Example of Application

Representation Tree

$\bullet\ Y_1 = Y$

$\delta_{C_1} \downarrow$

$\bullet\ Y_2 = X_1$

$\varepsilon_{C_2^t} \downarrow$

$\bullet\ Y_3 = X_2^*$

$W_2 = 11\underline{1}$

$\mathbf{Y}_2 = \{[10\underline{1}], [01\underline{0}]\}$

$\mathbf{Y}_2^* = \{[11\underline{0}], [01\underline{1}]\}$

Output of SEARCH_ALL $(\mathbf{Y}_2)$:

$\emptyset$.

Output of SEARCH_ALL $(\mathbf{Y}_2^*)$:

$C_2 = 1\underline{1}$

$\mathbf{Y}_3^* = \mathbf{X}_2 = \{[11\underline{0}]\}$

$\mathbf{Y}_3 = \mathbf{X}_2^* = \{[01\underline{0}], [10\underline{0}]\}$

$W_3 = W_2 \ominus C_2^t = 11\underline{0}$

(c)

Example of Application

Representation Tree

$\bullet\ Y_1 = Y$

$\delta_{C_1} \downarrow$

$\bullet\ Y_2 = X_1$

$\varepsilon_{C_2^t} \downarrow$

$\bullet\ Y_3 = X_2^*$

$\delta_{C_3} \downarrow$

$\bullet\ Y_4 = X_3$

$W_3 = 11\underline{0}$

$\mathbf{Y}_3 = \{[01\underline{0}], [10\underline{0}]\}$

$\mathbf{Y}_3^* = \{[11\underline{0}]\}$

Output of SEARCH_ALL $(\mathbf{Y}_3)$:

$C_3 = \underline{1}1$

$\mathbf{Y}_4 = \mathbf{X}_3 = \{[1\underline{0}]\}$

$W_4 = W_3 \ominus C_3^t = 1\underline{0}$

Output of SEARCH_ALL $(\mathbf{Y}_3^*)$:

$\emptyset$.

(d)

Example of Application

Representation Tree

$$Y_1 = Y \xrightarrow{\delta_{C_1}} Y_2 = X_1 \xrightarrow{\varepsilon_{C_2^t}} Y_3 = X_2^* \xrightarrow{\delta_{C_3}} Y_4 = X_3 \xrightarrow{\delta_{C_4}} Y_5$$

$W_4 = 1\underline{0}$

$\mathbf{Y}_4 = \{[1\underline{0}]\}$

$C_4 = \underline{0}1$

$\mathbf{Y}_5 = \{[\underline{1}]\}$

$W_5 = W_4 \ominus C_4^t = \underline{1}$

(e)

$$\psi = \epsilon_P \vee \epsilon_Q \vee \epsilon_R \vee \epsilon_S$$

9 basic operations

$$\psi = \delta_{C_1} \varepsilon_{C_2^t} \delta_{C_3} \delta_{C_4}$$

7 basic operations

Experimental Results

Operator	NOSD	NOBS	T
ψ_1	111	30	1.8s
ψ_2	111	16	12.0s
ψ_3	319	17	1.0s
ψ_4	431	20	4.2s
ψ_5	187	15	48m23s
ψ_6	143	16	30m43s

NOSD = Number of Operations using the Sup-Decomposition.

NOBS = Number of Operations using the Best Solution.

T = Time taken by the algorithm for finding the Best Solution.

Conclusions

- **Problem of transforming the sup-decomposition to sequential decompositions (when they exist).**

- **Applied to compute sequential decompositions of operators built by compositions of dilations and erosions.**

- **Minkowski Factorization Equation:**
    - **General case;**
    - **Fixed Right Extremities.**

- **Future step: Find parallel algorithms for these results.**