Design of Morphological Operators by Learning

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Introduction

Binary operator design: W-operators

Binary operator design: constraint W-operators

Gray-scale operator design: apertures

Gray-scale operator design: stack filters

Conclusion
Introduction
Morphological Machine (MMach)
Properties

- Any finite lattice operator can be implemented as a program of a MMach
- Finite lattices of practical importance are the lattice of binary and gray-scale images
Morphological Toolbox

- Library of hierarchical functions:
- primitives: elementary operators and operations;
- high order operators: primitives and high order operators
Heuristic Design

- **Divide** the problem in subproblems
- Each **subproblem** is solved by a **toolbox** function
- **Integrate** the subproblems solution
Automatic Design

- Operator learning in a standard representation
- Finding an equivalent and more efficient representation
Operator Design

1. Collect examples
2. Decision
3. Minimization

- a. Examples table
- b. Decision table
- c. Operator basis
Application

Noise Image \xrightarrow{\Psi} \text{Operator basis} \{[110,111], [101,111], [011,111]\} \xrightarrow{} \text{Restaured Image}
Change of Representation

\[ \epsilon_{A_1} \lor \epsilon_{A_2} \lor \cdots \lor \epsilon_{A_n} \]

Operators

Phrases

\[ \delta_A \epsilon_A \]

ψ
Operator Design
The problem

Find an image operator that transforms the observed image to the respective ideal (or “close to the ideal”) image.
Binary image operators

Binary image: \( f : E \to \{0, 1\} \)

Binary images can be understood as sets:
\[
\begin{align*}
  f & \longleftrightarrow S \\
  x \in S & \iff f(x) = 1 \quad \forall x \in E
\end{align*}
\]

\((\mathcal{P}(E), \subseteq)\) is a complete Boolean lattice

Binary image operators = set operators:
\[
\Psi : \mathcal{P}(E) \to \mathcal{P}(E)
\]
Translation invariance

Translation of $S$ by $z$:

$$S_z = \{ x + z : x \in S \}$$

$$\Psi : \mathcal{P}(E) \to \mathcal{P}(E)$$ is translation-invariant iff

$$\Psi(S_z) = [\Psi(S)]_z$$
Local definition

Window: \( W \subseteq E \)

An image operator is **locally defined** within \( W \) iff

\[
x \in \Psi(S) \iff x \in \Psi(S \cap W_x)
\]

\( S \cap W_x \) \hspace{1cm} \( \Psi(S \cap W_x) \)
**W-operators**

Translation invariance + local definition within = \( W \)-operators

\[ \Psi(S)(z) = \psi \]

\( W \)-operators are characterized by Boolean functions.
Representation

Window $W = 1 \times 3$

\[ \mathcal{K}(\Psi) = \{ \text{X11, X0, 11X} \} \]

\[ \mathcal{B}(\Psi) = \{ \text{X11, X0, 11X} \} \]

\[ \psi = \lambda_{X11} \cup \lambda_{1X0} \cup \lambda_{11X} \]
X and Y are jointly stationary

\[ P(S \cap W_z, Y) \]

is the same for any \( z \) in \( E \)
Stationary Process
Join Stationary Process
Error measure

- Design goal is to find a function with minimum risk.

- Risk (expected loss) of a function:

  $$R(\psi) = E[l(\psi(X), Y)]$$

- Loss function

  $$l : \{0, 1\} \times \{0, 1\} \to \mathbb{R}^+$$

- $X$ is a random set
- $Y$ is a binary random variable
MAE example

Example: MAE loss function

\[ l_{MAE}(a, b) = |a - b| \quad a, b \in \{0, 1\} \]

\[ MAE(\Psi) = E[|\psi(X) - Y|] \]

Optimal MAE function

\[ \psi(X) = \begin{cases} 
1 & p(1, X) > p(0, X) \\
0 & p(1, X) \leq p(0, X) 
\end{cases} \]
Design procedure

\[ X = S \cap W_x \quad Y = I(\varepsilon) \]
PAC learning

L is Probably Approximately Correct (PAC)

For $m > m(\varepsilon, \delta)$ examples

$\Pr(|R(\psi) - R(\psi_{opt})| < \varepsilon) > 1 - \delta$

$\varepsilon, \delta \in (0, 1)$
Edge detection

Training images

Test images
Noise filtering

Training images

Test images
Texture extraction (1)

Training images
Texture extraction (2)

Test images
Fracture Detection

Training images

Test images
Amount of data available

Noise 1 image 2 images 3 images 8 images 13 images 25 images 32 images
18.10% 3.13% 2.68% 2.21% 2.01% 1.82% 1.69% 1.60%
window 5x5, 6 training images

addition: 2%  
subtraction: 1%  
distinct patterns: 140.060  
in 1.548.384

addition: 3%  
subtraction: 3%  
distinct patterns: 266.743  
em 1.548.384

addition: 6%  
subtraction: 6%  
distinct patterns: 487.494  
in 1.548.384
Size of the window

Error rate in function of window size

- 3x3
- 4x4
- 5x5
- 6x6
- 7x7
Constraint Design
Constraints
Constraints

![Graph showing the relationship between sample size and error for different constraints.](image)

- $\varepsilon_{\text{des}}$
- $\varepsilon_{\text{des-con}}$
- $\varepsilon_{\text{opt}}$
- $\varepsilon_{\text{opt-con}}$

Sample size

- $N_1$
- $N_0$
- $N_2$
Structural Constraints

- impose maximum number of elements in the basis
- use alternative structural representations
  (e.g., sequential)
Algebraic constraints

- consider a class of operators satisfying a given algebraic property (e.g., increasingness, idempotence, auto-dualism, etc)

- consider a constraint by a multi-resolution criteria

- consider a constraint by an envelope of operators

- consider a constraint by a multi-resolution envelope criteria
Structural Constraint
Minimum number of intervals in the basis
42
Figure (a) shows a grid of black squares. Figure (b) is a graph plotting MAE against the number of structuring elements in the basis. The graph shows a decreasing trend as the number of elements increases. Figure (c) continues to display different configurations of black squares. The text below the graph reads: "ponent is a lar in the ction 2 this system for".
Iterative design
Motivation: composition of operators over small windows produces an operator over a larger window.

\[ \Psi = \Psi_2(\Psi_1) \] is a \( W \oplus W \)-operator.
Application example

reduzida de seu produ
clevados, juntamente ci
tos a adquirir o produ
quantidades produzidas
necessariamente, a red
que a empresa monopi
cado. Quantidades su

iteration 1

iteration 2
Application example

test image

iteration 1

iteration 2

iteration 3

iteration 4

iteration 5

iteration 6
Algebraic Constraint
Increasing \( W \)-operators

\[ x \leq y \Rightarrow \psi(x) \leq \psi(y) \]

Increasing

non-increasing
Design of increasing $W$-operators

- non-increasing
- increasing
Multi-resolution constraint
2 variables: $2^2 = 4$

4 variables: $2^4 = 16$

8 variables: $2^8 = 256$
\[
D_1 = P(W_1) \\
D_0 = P(W_0)
\]

\[
z_i = p_i(x_{i1}, \ldots, x_{i9}) , \quad z = p(x) , \quad p=(p_1, \ldots, p_9)
\]

Let \( \phi : D_1 \rightarrow \{0,1\} \), it defines the operator \( \Psi_\phi \) on \( D_0 \) by

\[
\Psi_\phi (x) = \phi(p(x))
\]

The operator \( \Psi_\phi \) is constrained by resolution to \( D_1 \)

**Equivalence classes defined by**

\[
p(x) = p(y)
\]
Properties

Operators over $D_0$

Operators over $D_1$ constrained by resolution on $D_1$

Operators over $D_0$
$D_2 = P(W_2), \ D_1 = P(W_1), \ D_0 = P(W_0)$

$x \in D_0, \ z \in D_1, \ v \in D_2.$

$z_i = p_i(x_{i,1}, \ldots, x_{i,9}), \ z = p(x), \quad p = (p_1, \ldots, p_{81})$

$v_i = w_i(x_{i,1}, \ldots, x_{i,81}), \ v = w(x), \quad w = (w_1, \ldots, w_9)$

The equivalence classes defined by $p$ may be different by the ones defined by $w.$
\( \phi(x) = \begin{cases} 
\psi_{0,N}(x) & \text{if } N(x) > 0 \\
\psi_{1,N}(x) & \text{if } N(x) = 0, N(\rho_1(x)) > 0 \\
\vdots \\
\psi_{m-1,N}(x) & \text{if } N(x) = 0, \ldots, N(\rho_{m-2}(x)) = 0, N(\rho_{m-1}(x)) > 0 \\
\psi_{m,N}(x) & \text{if } N(x) = 0, \ldots, N(\rho_{m-1}(x)) = 0, N(\rho_m(x)) > 0 
\end{cases} \)
Multiresolution Noise

image

noise

image + noise
3x3 window

Pyramid

Restauration

Persisting noise
The graph illustrates the relationship between the window sequence (W0, W1, W2, W3) and the mean squared error (MSE) for two different apertures: Multiresolution Aperture and Aperture. The Multiresolution Aperture shows a decreasing trend in MSE as the window sequence progresses, whereas the Aperture shows an increasing trend, peaking at W1 and then declining.
Example

Hybrid Multiresolution Filter: Experiment I

Window sequence

MAE

Single operator  Pyramid operator
Envelope constraint
Independent Constraints

Constraints

Restriction of the operators space

\[ K(\psi_{opt}) \in Q \subseteq P(P(W)) \]

Independent Constraint

Let be \( A, B \subseteq P(W) \) with \( A \subseteq B \):

\[ h_\psi(x) = 1 \ \forall \ x \in A \ \& \ h_\psi(x) = 0 \ \forall \ x \notin B, \]

\[ \forall \psi : K(\psi) \in Q \]
Independent Constraints

**Proposition:** if $Q$ is an independent restriction then exist a par of operators $(\alpha, \beta)$ such that, for any $\psi \in \Psi_w$

$$K(\psi) \in Q \iff \alpha \leq \psi \leq \beta$$

where $K(\alpha) = A$ and $K(\beta) = B$

- All independent constraint is characterized by two operators $\alpha$ and $\beta$
- The pair $(\alpha, \beta)$ is called "Envelope"
Definition

Desing of two heristic filters $\alpha$ and $\beta$ such that when we know that $\alpha \leq \psi_{opt} \leq \beta$, the restriction is defined by:

$$Q = \{ \psi : \alpha \leq \psi \leq \beta \}$$

and any filter $\psi$ can be projected into the restriction by

$$\psi_{env} = (\psi \lor \alpha) \land \beta$$
Definition

Operators over $P(W)$ constrained by envelope
Properties

♦ $(\psi_{opt} \lor \alpha) \land \beta$ is optimal in $Q$.

♦ If $\alpha \leq \psi_{opt} \leq \beta$ then $\text{Error}[\psi_{env}] \leq \text{Error}[\psi]$.

♦ If $\alpha \leq \psi_{opt} \leq \beta$ is not true, then

$$\lim_{N \to \infty} \text{Error}[\psi_{env,N}] > \lim_{N \to \infty} \text{Error}[\psi_N]$$
Example
Noise Edge Detection

Ground Image → Noise Addition → Noisy Image → Restoration → Filtered Image → Edge Detection → Edge Detected

Direct Edge Detection
**Restoration**  
\[ \text{a) Machine design of the restoration} \]

\[ \psi_{\text{pac}} \text{ designed by examples} \]

\[ \text{b) Human-machine design of the restoration} \]

\[ \psi_{\text{con}} = (\psi_{\text{pac}} \cap \beta) \cup \alpha \]

\[ \alpha = \delta_{B \oplus B} \epsilon_{B \oplus B} \delta_{B} \epsilon_{B} \text{ and } \beta = \epsilon_{B \oplus B} \delta_{B \oplus B} \epsilon_{B} \delta_{B} \]

\[ \alpha \text{ and } \beta \text{ are alternating sequential filters with} \]

\[ P[\alpha(S) \leq I \leq \beta(S)] \approx 1 \]

\[ B \text{ is the 3x3 square} \]

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<thead>
<tr>
<th>Machine design of the restoration</th>
<th>Human-Machine design of the restoration</th>
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<td>0.28 %</td>
<td>0.13 %</td>
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![Graph showing error comparison between machine and human-machine design of restoration](image-url)
Noise Edge Detection

Edge Detection

- **a**) Machine design over noisy images
  \[ \zeta_{\text{pac}} \] designed by examples from noisy images

- **b**) Human design after restoration
  \[ \zeta = I_d - \varepsilon_B \]
  \( B \) is the 3x3 square

- **c**) Machine design after restoration
  \[ \zeta_{\text{pac}} \] designed by examples from restored images

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<th>Machine design over noisy images</th>
<th>Human design after restoration</th>
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<td>0.65 %</td>
<td>0.27 %</td>
<td>0.24 %</td>
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![Graph showing error (%)](chart.png)
Noise Edge Detection

Machine design over noisy images
Error = 0.65%

Human design after restoration
Error = 0.27%

Machine design after restoration
Error = 0.24%
Envelope multi-resolution constraint
Definition

- $W_1 \subset W_0$, $\rho: D_0 \rightarrow D_1$ is a resolution mapping
- $\alpha, \beta: D_1 \rightarrow \{0, 1\}$ with $\alpha \leq \beta$
- $\psi: D_0 \rightarrow \{0, 1\}$

$$
\psi_\rho(x) = \begin{cases} 
1 & \text{if } \alpha(\rho(x)) = 1 \\
0 & \text{if } \beta(\rho(x)) = 0 \\
\psi(x) & \text{otherwise}
\end{cases}
$$

- $\psi_\rho = (\psi \land \beta') \lor \alpha'$, $\alpha'(x) = \alpha(\rho(x))$ and $\beta'(x) = \beta(\rho(x))$
Definition
Teorema

\[ \beta' \]  
\[ \psi \]  
\[ \psi_\rho \]  
\[ Q \]  
\[ \alpha' \]  
\[ D_0 \]  
\[ \beta \]  
\[ \phi \]  
\[ \phi_{env} \]  
\[ \alpha \]  
\[ D_1 \]
Piramidal Design:

Let $\psi_{i,\text{env},N} = (\psi_{i,N} \wedge \beta') \vee \alpha'$, be the projection of the resolution constrained filter inside the envelope $(\alpha', \beta')$

$$
\psi_{\text{env-mres}}(x) = \begin{cases} 
\psi_{0,N}(x) & \text{if } N(x) > 0 \\
\psi_{1,\text{env},N}(x) & \text{if } N(x) = 0, N(\rho_1(x)) > 0 \\
\vdots & \\
\psi_{m-1,\text{env},N}(x) & \text{if } N(x) = 0, \ldots, N(\rho_{m-2}(x)) = 0, N(\rho_{m-1}(x)) > 0 \\
\psi_{m,\text{env},N}(x) & \text{if } N(x) = 0, \ldots, N(\rho_{m-1}(x)) = 0, N(\rho_m(x)) > 0
\end{cases}
$$
Properties:

♦ $\psi_{\text{env-mres}}$ is a consistent estimator of $\psi_{\text{opt}}$

♦ If the envelope is well defined on $D_1$, then the $\rho$-envelope of a resolution constrained filter is advantageous
Figura 5.7: Primeira imagem corrompida com ruído

$\alpha = \varepsilon_B(\gamma_E \phi_E \gamma_E)$

$\beta = \delta_B(\phi_E \gamma_E \phi_E)$
Gray-scale operator design: aperture
Spatial Translation Invariance

\[ f_t(x) = f(x-t) \]

\[ \Psi(f)(0) = \Psi(f_t)(t) = \Psi_t(f)(0) \]

\[ \Psi_t(f)(x) = \Psi(f)(x-t) \]
Gray-scale Translation Invariance

\[(f+h)(x) = f(x) + h\]

\[\Psi(f)(x) = \Psi \cdot f(x) \]

\[\Psi(f+h)(x) = \Psi(f)(x) + h\]
Locally defined in $W$

$$\Psi(f)(x) = \Psi(f/W_x)(x)$$
Locally defined in $W$ and $K$

$$(u / K_y)(z) = \land \{ \lor \{-k, u(z) - y\}, k \}$$
Aperture Operator

\[ W \]

\[ K = \{-2, -1, 0, 1, 2\} \]
Aperture Operator

\[
\beta_\psi
\]

\[
\psi
\]

\[
\mathbf{u}(o)
\]

\[
\beta_\psi
\]

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- Let $a, b \in \text{Fun}[W,L]$, $a \leq b$ iff $a(x) \leq b(x)$, $x \in W$

- Interval $[a,b] = \{u \in \text{Fun}[W,L] : a \leq u \leq b\}$
Sup-generating operator:

\[ \lambda_{a,b}(u) = 1 \iff u \in [a, b] \]
Kernel of $\psi$ at $y$: $K(\psi)(y) = \{ u \in \text{Fun}[W,L] : y \leq \psi(u) \}$
Basis of $\psi$ at $y$: $B(\psi)$ is the set of maximal intervals contained in $K(\psi)$
\[ \psi(u) = \bigcup \{ y \in M : \bigcup \{ \lambda_{a,b}(u) : [a,b] \in B(\psi)(y) \} = 1 \} \]

\[ \psi(-1,-1) = 1 \]
These are part of the observed and ideal images (512x512)
MAE x Number of Examples

![Graph showing MAE error (%) against number of examples for different filter sizes.](image)
Deblurring - Aperture x Optimal linear

Aperture 17p x 5 x 5

Optimal linear 7x7
Resolution Enhancement
Resolution Enhancement

(0,0)

Ψ₀

Ψ₁

Ψ₂

Ψ₃

(0,1)
Resolution Enhancement - Results

Original

Aperture: 3x3x21x51

Linear

Bilinear
Resolution Enhancement - Results

Zoom

Original

Aperture: 3x3x21x51

Linear

Bilinear
Gray-scale operator design: stack filters
A stack filter is a gray-scale operator characterized by a positive (i.e., increasing) Boolean function

\[ \psi(f) = \max\{t \in K : \psi(T_t[f]) = 1\} \]

where

\[ T_t[f] = \{x \in W : f(x) \geq t\} \]
Impulse noise removal (1)

training images
Impulse noise removal (2)

test image

iteration 1
Impulse noise removal (3)

test image

iteration 5
Robustness (1)

test image

iteration 1
Robustness (2)

test image

iteration 5
Conclusion

- Design of Morphological Operators: a discrete nature problem
- Fundamentals: Algebra, Statistics, Combinatory
- Real problems solution
- Design techniques adequate to introduce prior knowledge
- Identification of Lattice Dynamical Systems