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Modeling Tactical Changes of Formation in Association Football as a Zero-Sum Game

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Modeling Tactical Changes of Formation in Association Football as a Zero-Sum Game*

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Abstract

Although tactical decisions made by managers during a match of team sports are very important, there have been few quantitative analyses which include the effect of interaction between both teams' decisions, because of the complexity of the problem where one team's decision will affect the other team's. A game theoretic approach can be useful for tackling this type of problem.

This paper proposes a game theoretic approach to modeling tactical changes of formation in an association football match. We assume probabilities of scoring and conceding a goal follow Poisson distributions and use a regression model to evaluate the means of the distributions. These means represent the offensive strength for scoring a goal and defensive propensity to concede a goal in terms of a team's formation, i.e. a combination of the number of each type of outfield player on the pitch, and are estimated by means of the maximum likelihood method. We then develop a mathematical formulation with which we can calculate the probability of the home team winning the match, and use it to analyse tactical changes of the teams' formations, modeling the football match as a zero-sum game, in which the gain in probability of one team winning is equal to the loss in probability of the other team winning. We demonstrate how the managers' decisions affect the probability of winning the match using real data of the Japan professional football league, by showing four cases of the quality of both managers' decisions, depending on whether they each use their best or worst strategies.

There still remains some uncertainty and longer observational studies will be required for a complete analysis, but this method can help to evaluate quantitatively the quality of tactical decisions made by managers.

KEYWORDS: football, formation, game theory, tactics, zero-sum game

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1. Introduction

In team sports, tactical decisions made by managers during a match are very important factors which often affect the result of the match significantly. In general, it is not easy for managers to make their decisions, because one team's decision will affect the other team's, i.e. there is an interaction between the teams' decisions. A game theoretic approach can be useful for formulating and solving this type of problem.

Until now game theory has been applied mainly to economic issues rather than to sports, though some applications to sports such as association football (Sadovskii and Sadovskii, 1993) (henceforth simply referred to as football although it is also known as soccer) and tennis (Winston, 1993) have been used in textbooks to illustrate the concepts of game theory. In terms of academic research, the only application to sports appears to be that of Yoshida et al. (1994), who applied game theory to volleyball in order to identify the optimal strategy of the use of the four blocking formations, conducting an experiment using intercollegiate players by repeating the attacking-blocking situation. For football, Cowan (1992) made an basic analysis of allocation of offensive and defensive players, but his research is just an analytical formulation without any real match data. There does not appear to have been any research published applying game theory to football using real match data.

In terms of modelling a football match, Hirotsu and Wright (2002, 2003a, 2003b) use a Markov process model, analysing the propensity not only to score or concede goals, but also to gain or lose possession. They demonstrated how their approach may help to determine the best strategy for changing the configuration of a team, using dynamic programming. Wright and Hirotsu (2003) also studied the use of the professional foul and discussed the drawbacks of the current system for punishing such fouls. Although such research to some extent examines tactical decisions in a football match, the interaction between the two teams' tactical decisions has not until now been considered.

In this paper, therefore, we extend the model of a football match to take account of the interaction between two teams' tactical decisions using game theory. The tactical decisions we are studying involve the change of configuration of a team by substitutions or other activities. We test this model on real data of the Japan professional football league (J. League) for the 2002 season. Here we focus on two well-known teams in the J. League and analyse their offensive and defensive strengths based on their formation i.e. a combination of the number of each type of outfield player on the pitch. Using these strengths, we then quantitatively develop the mathematical formulation for analysing tactical changes of the teams' formations, and demonstrate how the managers' decisions affect the probability of winning the match.

2. The Poisson Regression Model

A comprehensive mathematical model describing the progress of a football match has been developed using Poisson regression. Maher (1982) claimed to show that a Poisson distribution may be appropriate to model goal-scoring, taking account of differing team strengths. He shows that a relatively simple Poisson regression model gives a reasonably good fit to data obtained from the four English Football League Divisions from the 1971-2 to the 1973-4 season.

Maher's Poisson regression model was used and developed by other researchers: Lee (1997) analysed the strengths of the English Premier League teams based on the results of the 1995-96 season; Dixon and Coles (1997) and Dixon and Robinson (1998) also evaluated the strengths of teams and built a more complicated model including time-dependency for making a profit in the betting market; and Dyte and Clarke (2000) also used a Poisson regression model for their analysis of the 1998 World Cup. Hirotsu and Wright (2002, 2003a, 2003b) looked beyond goal-scoring and incorporated factors for the gaining and losing of possession into the model. However, here we develop Lee's model so as to avoid making the model too complicated, in order to demonstrate how game theory can be applied.

In Lee's model, the number of goals scored by a home team is assumed to follow a Poisson distribution with mean λ, and similarly the number scored by an away team follows a Poisson distribution with mean μ. The parameters, λ and μ, can be used to reflect the value of home advantage and the qualities of both teams, by assuming that they are an amalgam of a number of factors such that

$$\begin{aligned} \lambda &= \alpha \cdot \alpha_{home} \cdot \alpha_{score}(A) \cdot \alpha_{concede}(B) \\ \mu &= \alpha \cdot \alpha_{score}(B) \cdot \alpha_{concede}(A) \end{aligned} \quad (1)$$

Here, α and α_{home} represent the intercept and home advantage related to scoring goals, α_{score}(A) and α_{score}(B) represent the offensive strength of team A at home and of team B away for scoring goals, respectively. α_{concede}(A) and α_{concede}(B) represent the tendency for team A at home and team B away to concede goals, respectively. Here, if the tendency to concede goals is large, the team's defensive strength is thought to be small. The model thus takes account of general home advantage, and incorporates the factors for scoring and conceding a goal for each team. The reason that Lee uses just a single home advantage parameter, rather than separating parameters for home scoring and home defending, may come from Maher's (1982) research, in which just a single home advantage parameter is found to be the most appropriate.

In this model, for example, if team A plays team B at team A's home, λ is expressed by, $\alpha \cdot \alpha_{home} \cdot \alpha_{score}(A) \cdot \alpha_{concede}(B)$ and if team A plays team B away μ is expressed by $\alpha \cdot \alpha_{score}(A) \cdot \alpha_{concede}(B)$. Thus the factor of home advantage is expressed by $\lambda / \mu = \alpha_{\text{home}}$. According to Lee' analysis, this value was estimated as 1.42, which reflects the fact that generally a team would be likely on average to score 1.42 as many goals at home as away.

We here redefine the above parameters, λ and μ, as the number of goals scored per minute by dividing them by 90, and we refer to λ and μ as scoring rates later on this paper. Practically, it is done by redefining $\alpha / 90$ as α in (1). As all of the parameters in (1) are positive in this situation, the logarithm of the mean is expressed by a linear combination of a number of factors such that

$$\log(\lambda) = \beta + \beta_{home} + \beta_{score}(A) + \beta_{concede}(B)$$
$$\log(\mu) = \beta + \beta_{score}(B) + \beta_{concede}(A) \ . \qquad (2)$$

Here, such a relation as $\beta = \log(\alpha)$ holds between (1) and (2). This model is called a generalized linear model in statistics. The values of these parameters can be estimated using the maximum likelihood method under the assumption of independence of the distributions which is found to be appropriate according to Lee (1997). If we know the number of goals scored and the total time played in the match, we can estimate the scoring rates of the match such that $\lambda = G_H / T$ and $\mu = G_A / T$, where G_H and G_A are the numbers of goals scored by a home team and an away team in a match, respectively. T is the total time played in the match. (The standard length of a match is 90 minutes, but there are often a few minutes of "stoppage time" added by the referee, in which case T will be more than 90 minutes.)

3. Modelling as a Zero-sum Game

3.1. The formation of a team

We now describe how to use the offensive and defensive strengths obtained from real data in order to model a football match as a zero-sum game, in which the sum of the gain of both teams is always zero. In the case of a football match, both teams cannot win at the same time - if one team wins the other team loses, and the gain of the probability of one team winning is equal to the loss of the probability of the other team winning.

Draws of course are possible in football. When we use the term "probability of winning" we actually mean "expected number of wins", where a draw is considered to be half a win and there are thus three possible numbers of wins for a team in a match: 1, ½ or 0. However, for simplicity, we will continue to use the term "probability of winning".

As strategies in the game, we focus on the tactical changes of formations of teams. We take into account the strength of the combination of different *types* of player rather than individuals. This is because there are not enough matches played in a football league, as discussed by Hirotsu and Wright (2003b), to provide sufficient data to estimate the offensive and defensive strengths of specific combinations of individual players.

In football, traditionally every outfield player (i.e. every player who is not a goalkeeper) can be categorized into one of three positions: defender, midfielder or striker. The different combinations of players on the pitch who fall into these categories are defined as formations. For example, the formation 4-4-2 refers to a situation with four defenders, four midfielders and two strikers. The selection of a formation is one of the main tactics employed by managers for the purpose of making the team play more or less offensively or defensively (see e.g. Bangsbo and Peitersen, 2000).

3.2. Game theoretic formulation of a football match

Now we explain the modelling of a football match including both teams' tactics concerning changes of their formations. First we assume that a change of formation is made by a substitution of an outfield player. We will discuss the case where the change of formation is not made by a substitution (i.e. it is reversible) in Section 5. Further, for simplicity, substitution between the same type of players is not counted. That is, in a real match, a player may be substituted by another player of the same type without changing the number of each type of outfield player on the pitch, for a variety of possible reasons such as tiredness, injury or poor play, but this is not considered here.

Let $P^{sw}(r,t,f,g)$ be the probability of a home team winning a match with time t minutes remaining when the team leads by r goals, in formations f taken by a home team and g by an away team, if s substitutes are still available for the home team and w substitutes for the away team (s, w = 0, 1, 2 or 3). The probability of the home team scoring and conceding a goal in the next small time dt is expressed by $\lambda\, dt$ and $\mu\, dt$, respectively. The probability of the match remaining at the same score is thus $1-(\lambda+\mu)dt$. At first, considering the case where no substitutes are available (i.e. s, w = 0) in the remaining of the match, the probability of a home team winning a match with time $t+dt$ minutes remaining when the team leads by r goals in formations f taken by a home team and g by an away team is expressed by $P^{00}(r,t+dt,f,g)$. The lead of r goals will be r+1 goals with t minutes remaining with probability $\lambda\, dt$ if the home team scores a goal in the next small time dt. This lead will be r-1 goals with probability $\mu\, dt$ if the away team scores a goal. Otherwise, if neither team scores, this lead remains r goals, with probability $1-(\lambda+\mu)\, dt$. As these three cases are mutually exclusive, $P^{00}(r,t+dt,f,g)$ can be expressed by the sum of these three terms as follows:

$$
\begin{aligned}
&P^{00}(r,t+dt,f,g) \\
&=P^{00}(r+1,t,f,g)\cdot\lambda(f,g)dt+P^{00}(r-1,t,f,g))\cdot\mu(f,g)dt+P^{00}(r,t,f,g)\cdot\{1-(\lambda(f,g)+\mu(f,g))dt\}
\end{aligned}
\tag{3}
$$

Here, note that the scoring rates, λ and μ, depend on the formations taken by both teams. The scoring rates are thus explicitly shown as functions of f and g in expression (3).

In the case where $s, w \geqq 1$, both teams can make a substitution to change their formation in the next small time dt with time t minutes remaining. Hence, we look at the following four possible cases separately in this situation:

- No teams make a substitution;
- Only the home team makes a substitution;
- Only the away team make a substitution;
- Both teams make a substitution.

If neither team makes a substitution in this situation, the probabilities of the home team scoring a goal, conceding a goal or neither in the next small time dt are $\lambda\, dt$, $\mu\, dt$ and $1-(\lambda+\mu)\, dt$, respectively. Thus:

$$P^{sw}(r,t+dt,f,g)$$
$$= P^{sw}(r+1,t,f,g)\cdot\lambda(f,g)dt + P^{sw}(r-1,t,f,g))\cdot\mu(f,g)dt + P^{sw}(r,t,f,g)\cdot\{1-(\lambda(f,g)+\mu(f,g))dt\} \quad (4)$$

On the other hand, if only the home team makes a substitution to change its formation from *f* to *f'* in this situation, the number of available substitutes *s* decreases to *s'*, which depends on the change of formation. For instance, *s'* = *s* - 1 in the case from *f* = 4-4-2 to *f'* = 5-3-2, because only one substitution is necessary to bring about this formation change. Thus:

$$P^{sw}(r,t+dt,f,g)$$
$$= P^{s'w}(r+1,t,f',g)\cdot\lambda(f',g)dt + P^{s'w}(r-1,t,f',g))\cdot\mu(f',g)dt + P^{s'w}(r,t,f',g)\cdot\{1-(\lambda(f',g)+\mu(f',g))dt\} \quad (5)$$

In a similar manner we can derive the equations corresponding to the cases where only the away team makes a substitution and where both teams make a substitution. We summarize the above four cases of the probability of the home team winning in the situation in Table 1. Since there are possibly other changes of formation such as from *f* to *f''*, *f'''*,⋯ or from *g* to *g''*, *g '''*,⋯, we define the tactics *i* (=0,1,2, ・・・) and *j* (=0,1,2, ・・・) for both teams such that tactic *i* =0, 1, 2, ⋯ corresponds to no change, the change from *f* to *f'*, the change from *f* to *f''*, ⋯, respectively.

Table 1 Probability of home team winning the match allowing for changes in formation.

$P^{sw}(r,t+dt,f,g)$			Away team: Tactic 0 — No change ($g \to g$)	Away team: Tactic 1 — Change ($g \to g'$)
Home team	Tactic 0	No change ($f \to f$)	P^{sw}(r+1,t,f,g) λ (f,g)dt + P^{sw}(r-1,t,f,g) μ (f,g)dt + P^{sw}(r,t,f,g) {1-(λ (f,g) + μ (f,g))dt}	$P^{sw'}$(r+1,t,f,g') λ (f,g')dt + $P^{sw'}$(r-1,t,f,g') μ (f,g')dt + $P^{sw'}$(r,t,f,g') {1-(λ (f,g') + μ (f,g'))dt}
Home team	Tactic 1	Change ($f \to f'$)	$P^{s'w}$(r+1,t,f',g) λ (f',g)dt + $P^{s'w}$(r-1,t,f',g) μ (f',g)dt + $P^{s'w}$(r,t,f',g) {1-(λ (f',g) + μ (f',g))dt}	$P^{s'w'}$(r+1,t,f',g') λ (f',g')dt + $P^{s'w'}$(r-1,t,f',g') μ (f',g')dt + $P^{s'w'}$(r,t,f',g') {1-(λ (f',g') + μ (f',g'))dt}

Using the above probabilities depending on the combination of both teams' decisions in the small time *dt* from time (*t* + *dt*) minutes remaining to *t* minutes remaining, we can formulate this problem as a zero-sum game. In a zero-sum game, each team chooses a tactic which enables the team to get the best it can. We show an example of how each team chooses its tactic following the manner of game theory in the Appendix. Here, if this game has a saddle point shown in the Appendix, the following equation holds:

$$
P^{sw}(r,t+dt,f,g)=
$$
$$
\max_i \min_j \begin{cases}
P^{sw}(r+1,t,f,g)\cdot\lambda(f,g)dt+P^{sw}(r-1,t,f,g)\cdot\mu(f,g)dt+P^{sw}(r,t,f,g)\cdot\{1-((\lambda(f,g)+\mu(f,g))dt\} \\
\quad :\text{Tactics i=0 and j=0 (no substitution at time t remaining)} \\
P^{s'w}(r+1,t,f',g)\cdot\lambda(f',g)dt+P^{s'w}(r-1,t,f',g)\cdot\mu(f',g)dt+P^{s'w}(r,t,f',g)\cdot\{1-(\lambda(f',g)+\mu(f',g))dt\} \\
\quad :\text{Tactics i=1 and j=0 (now making the substitution } (f\to f') \text{ at time t remaining)} \\
P^{sw'}(r+1,t,f,g')\cdot\lambda(f,g')dt+P^{sw'}(r-1,t,f,g')\cdot\mu(f,g')dt+P^{sw'}(r,t,f,g')\cdot\{1-(\lambda(f,g')+\mu(f,g'))dt\} \\
\quad :\text{Tactics i=0 and j=1 (now making the substitution } (g\to g') \text{ at time t remaining)} \\
P^{s'w'}(r+1,t,f',g')\cdot\lambda(f',g')dt+P^{s'w'}(r-1,t,f',g')\cdot\mu(f',g')dt+P^{s'w'}(r,t,f',g')\cdot\{1-(\lambda(f',g')+\mu(f',g'))dt\} \\
\quad :\text{Tactics i=1 and j=1 (now making the substitution } (f\to f') \text{ and } (g\to g') \text{ at time t remaining)} \\
\quad \cdots\cdots
\end{cases} \tag{6}
$$

where maximisation is taken from possible different tactics i (=0,1,2, $\cdots$) for the home team and minimisation is from possible different tactics j (=0,1,2, $\cdots$) for the away team, although we just show it explicitly the case that i, j = 0, 1.

Here, each team chooses its best decisions concerning formation change, in the sense that it makes a substitution with the optimal type of substitute at the optimum time so as to maximise the probability of its winning the match. The order of the timing of the substitution of the home team and the away team does not matter for the operation of " $\max_i \min_j$ " in (6), because the result of operating " $\max_i \min_j$ " and " $\min_j \max_i$ " is same in the case where a saddle point exists, as discussed in the Appendix.

This recursive equation in the form of a zero-sum game can be solved with the following boundary conditions: $P^{sw}(r,0,i,j)$ = 1 if r > 0, 0.5 if r = 0 and 0 if r < 0, in terms of any formations. Here, we note that the total amount of probability is 1, and each team aims to maximize its own probability.

Using this type of formulation, we can also obtain the probability of winning when the home team always makes its best decision and the away team always makes its worst decision, or vice versa, by taking " $\max_i \max_j$ " or " $\min_i \min_j$ " in (6). Moreover, by taking " $\min_i \max_j$ " in (6) we can get the case where both teams always make their worst decisions. In the next section, we demonstrate how the managers' decisions affect the probabilities of winning the match, by showing these four cases, using real data of the J. League.

4. A Numerical Example

4.1. Sample Data

We now present a numerical example of our procedure to calculate the probability of winning a match depending on the quality of both teams' tactical decisions. Here, we use the data of score and played time of each match of the J. League Division 1 in the 2002 season; a small subset of this data is shown below in Table 2. For this numerical example, we have chosen two well-known J. League teams, Yokohama F Marinos and Kashima Antlers, out of the 16 teams in the division. In order to identify the formation of both teams, we use the data of players' registered position for each match, and we then use the number of minutes played with each formation.

Table 2 Subset of the data of the 2002 season of the J. League

Date	Home	Away	Score G_H	Score G_A	T (min)
2 Mar. 02	F.C. Tokyo	Kashima Antlers (4-4-2)	4	0	62
		Kashima Antlers (4-5-1)	0	2	28
2 Mar. 02	Jubilo Iwata	Nagoya Grampus Eight	2	0	90
3 Mar. 02	Gamba Osakla	Kashiwa Reysol	1	0	90
3 Mar. 02	JEF United Ichihara	Kyoto Purple Sanga	2	1	90
3 Mar. 02	Sanfrecce Hiroshima	Consadole Sapporo	5	1	90
3 Mar. 02	Shimizu S-Pulse	Vissel Kobe	1	0	93
3 Mar. 02	Vegalta Sendai	Tokyo Verdy 1969	1	0	90
3 Mar. 02	Yokohama F Marinos (4-4-2)	Urawa Reds	1	0	42
	Yokohama F Marinos (3-4-2)		0	0	39
	Yokohama F Marinos (4-3-2)		0	0	9
…	…	…	…	…	…
30 Nov. 02	Urawa Reds	Yokohama F Marinos (4-4-2)	0	1	90
30 Nov. 02	Kyoto Purple Sanga	JEF United Ichihara	3	2	90
30 Nov. 02	Consadole Sapporo	Sanfrecce Hiroshima	5	4	99
30 Nov. 02	Kashima Antlers (4-4-2)	F.C. Tokyo	1	1	70
	Kashima Antlers (3-4-2)		0	0	1
	Kashima Antlers (4-3-2)		0	0	19
30 Nov. 02	Tokyo Verdy 1969	Vegalta Sendai	3	1	90
30 Nov. 02	Kashiwa Reysol	Gamba Osakla	2	0	90
30 Nov. 02	Vissel Kobe	Shimizu S-Pulse	3	0	90
30 Nov. 02	Nagoya Grampus Eight	Jubilo Iwata	2	3	90

4.2. Estimation of the strength of the formation

Based on the data in Table 2, we then estimate the offensive and defensive strengths of all formations of these two teams and the strengths of all the other 14 teams (without considering their formations), by means of the maximum likelihood method. Using this method we can make a macro-evaluation of strengths of the formation. In practice, the Poisson regression model in expression (2) is applied to the data, and the parameters, β, β_{home}, β_{score} and $\beta_{concede}$ are estimated under the assumption that the number of goals scored by a team or a formation of the team follows an independent Poisson distribution. That is, in expression (2) we use the scoring rates of $\lambda = G_H / T$ and $\mu = G_A / T$, where the values of G_H, G_A and T are based on a team or a formation in the match as shown in Table 2. In the calculation for the estimation, the Newton-Raphson method was used on the GLIM (Generalised Linear Interactive Modelling) statistical package (Francis et al., 1993).

According to the data, Yokohama FM had three main alternative formations and Kashima A had two main formations. The main formations for Yokohama FM were 3-5-2, 4-4-2 and 4-3-3, which were used for 1879, 450 and 326 minutes respectively, accounting for 91% of its total played time over the season. The main formations for Kashima A were 4-4-2 and 4-5-1, which were used for 1870 and 415 minutes respectively, accounting for 83% of its total played time.

As the results of the estimation, we obtain the offensive and defensive strengths of these formations as shown in Figure 1. In this figure these estimates indicate the propensity for scoring and conceding goals. Here, note that the direction of the y-axis representing defensive strength is reversed, because if the value is negatively larger, it means stronger in the sense of conceding fewer goals. Thus, formations plotted in the upper right area are relatively stronger than formations located in the lower left area. We also obtain β = -4.35 (0.23), β_{home} = 0.250 (0.078) (with standard errors in parentheses). This means that an average team is expected to score exp(-4.35) = 0.0129 goals per minute (1.16 per match) away and it will score 0.0166 (=0.0129×exp(0.250)) goals per minute (1.49 per match) at home.

As shown in Figure 1, formation 4-3-3 of Yokohama FM appears to be the best for scoring goals but worst for conceding goals; 4-4-2 is the best for not conceding goals; and 3-5-2 is the worst for scoring goals. For Kashima A, on the other hand, formation 4-5-1 appears to be better than 4-4-2 for scoring goals but worse for conceding goals. Taking into account both aspects of scoring and conceding a goal, it may well be the case that different formations are best in different circumstances, in terms of the score in the match and the amount of time remaining.

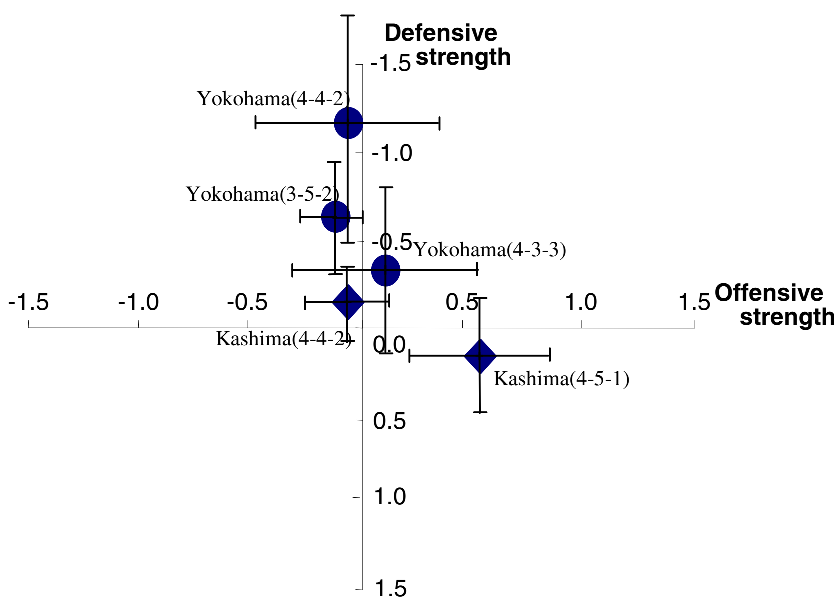


Figure 1 Offensive and defensive strengths for scoring a goal with standard errors.

In figure 1, the offensive strengths of formations 4-4-2 and 4-3-3 of Yokohama FM, for example, are difficult to distinguish significantly. As a further study, we could study the effect of the uncertainty of the estimated value of the strengths on the outcomes by using sensitivity analysis or another method. Although some factors of the estimates are not significantly different, it depends on the managers' belief about the estimates and practically the game theoretic approach can be used by setting the estimates according to the managers' intuition.

4.3. The substitution strategy in a home match

We now try to analyse both teams' substitution strategies for the change of formations in Yokohama FM's home match. The offensive and defensive strengths shown in Figure 1 are used to produce estimates of the scoring rates of the match, using expression (2) in reverse. That is, we put the values of β, β_{home}, β_{score}, and β_{concede}, in Figure 1 into expression (2) to estimate λ and μ. The estimates of the scoring rates of the match, depending on both teams' formation, are tabulated in Table 3.

Table 3 Estimates of scoring rates in Yokohama FM's home match

(λ, μ)		Formation of Kashima A	
		4-4-2	4-5-1
Formation of Yokohama FM	4-4-2	(0.0139, 0.0038)	(0.0190, 0.0069)
	4-3-3	(0.0159, 0.0087)	(0.0216, 0.0160)
	3-5-2	(0.0131, 0.0067)	(0.0179, 0.0123)

(unit: goals per minute)

To begin with, using the above values of scoring rates for each combination of their formations, we calculate the probability of Yokohama FM winning the match in the case where no substitutions are available following expression (3). In the calculation, the number of goals by which either team leads is assumed not to exceed 10 i.e. $-10 \leqq r \leqq 10$. The results of the calculations without any substitutions being available are tabulated in Table 4. Here ,Yokohama FM is assumed to be a home team, the remaining time is 90 minutes (t = 90) and the scores are level (r = 0) at the beginning of the match. Thus, the probability of Yokohama FM winning this match is represented by $P^{00}(0,90,f,g)$ (the number of substitutes available is zeros (s, w = 0)). As shown in the table, a saddle point of these probabilities is found in the combination of formation 4-4-2 for both teams with the value of 0.751. That is, if both teams make the best decision for starting formations without any formation changes being available, they will both choose 4-4-2 as an equilibrium point. On the other hand, if Yokohama FM chooses formation 4-4-2 and Kashima A chooses 4-5-1, this combination provides the best benefit for Yokohama FM and the worst for Kashima A. Further, the combination of 3-5-2 of Yokohama FM and 4-5-1 of Kashima A provides the worst for Yokohama FM and the best for Kashima A, while the combination of 3-5-2 of Yokohama FM and 4-4-2 of Kashima A is the equilibrium point in the sense of the worst choice for both teams.

Table 4 Probability of Yokohama FM winning the match at home without any substitutions.

$P^{00}(0,90,f,g)$		Formation of Kashima A	
		4-4-2	4-5-1
Formation of Yokohama FM	4-4-2	0.751	0.753
	4-3-3	0.660	0.605
	3-5-2	0.656	0.613

Now we introduce the change of formation by substitution. Firstly we allow only Yokohama FM to change its formation three times by means of substitution during the match. We can calculate the probability of Yokohama FM winning using expression (6) such that $s = 3$ and $w = 0$. Here, we note that in the J. League the maximum number of substitutions is three in 90 minutes. (Actually, there was the extra 30 minutes rule in the J. League in the 2002 season. That is, if the scores were level at 90 minutes, the game entered an extra 30 minutes and continued until either team scored. But since the 2003 season this extra 30 minutes rule has been abolished.)

In the case where both teams are allowed to change their formation 3 times by substitutions, the probability of Yokohama FM winning is calculated using expression (6). The calculation results of the probability of winning, depending on the starting formation, are shown in Table 5. As shown in the table, the probabilities of Yokohama FM winning are between the minimum and the maximum of the element in Table 4. The saddle point appears in the combination of 4-4-2 for both teams. That is, if both teams make the best decision for the changes of their formations during the match, they will choose 4-4-2 as starting formation.

Table 5 Probability of Yokohama FM winning the match at home in different starting formations in the case where both teams have 3 substitutes being available.

$P^{33}(0,90,f,g)$		Formation of Kashima A	
		4-4-2	4-5-1
Formation of Yokohama FM	4-4-2	0.73438	0.73451
	4-3-3	0.73437	0.73450
	3-5-2	0.73437	0.73450

We show a little more detail of the change of formations during the match after they start off with their best formation 4-4-2, under the condition that they always make their best decisions. Figure 2 shows a whole image of the flow of their best decisions of changing the formation by means of substitutions.

Number of substitutions remaining for Yokohama FM \ Number of substitutions remaining for Kashima A	3		2		1		0
3	(4-4-2, 4-4-2)	→	(4-4-2, 4-5-1)	→	(4-4-2, 4-4-2)	→	(4-4-2, 4-5-1)
	↓				↓		↓
2	(4-3-3, 4-4-2)				(4-3-3, 4-4-2)		(4-3-3, 4-5-1)
	↓				↓		↓
1	(4-4-2, 4-4-2)	→	(4-4-2, 4-5-1)	→	(4-4-2, 4-4-2)	→	(4-4-2, 4-5-1)
	↓				↓		↓
0	(4-3-3, 4-4-2)	→	(4-3-3, 4-5-1)	→	(4-3-3, 4-4-2)	→	(4-3-3, 4-5-1)

Figure 2 Whole image of the flow of best decisions of changing the formation depending on the number of substitutions available. (“ (f, g) ” such as “(4-4-2, 4-5-1)” represents the best formation of Yokohama FM and Kashima A, respectively.)

As there is not enough space to describe all the above decisions here, we just follow a part of the flow of the best decisions with the best timing of the substitution, representing the case where the number of goals by which either team leads is not to exceed 2, as shown in Figure 3. In the top block (3-3) of this figure, both teams start off with 4-4-2, with 3 substitutions available. Then, once Yokohama FM leads, Kashima A should make a substitution to change the formation from 4-4-2 to 4-5-1. Otherwise, if Yokohama FM falls behind by 2 goals with less than 52 minutes remaining, or by 1 goal with less than 32 minutes remaining, it should make a substitution to change the formation from 4-4-2 to 4-3-3. This result looks reasonable because 4-5-1 for Kashima A and 4-3-3 for Yokohama FM are the most offensive formations for scoring goals in order to get the scores level after falling behind, and 4-4-2 for both teams is the best for defending against conceding goals once it leads, as shown in Figure 1.

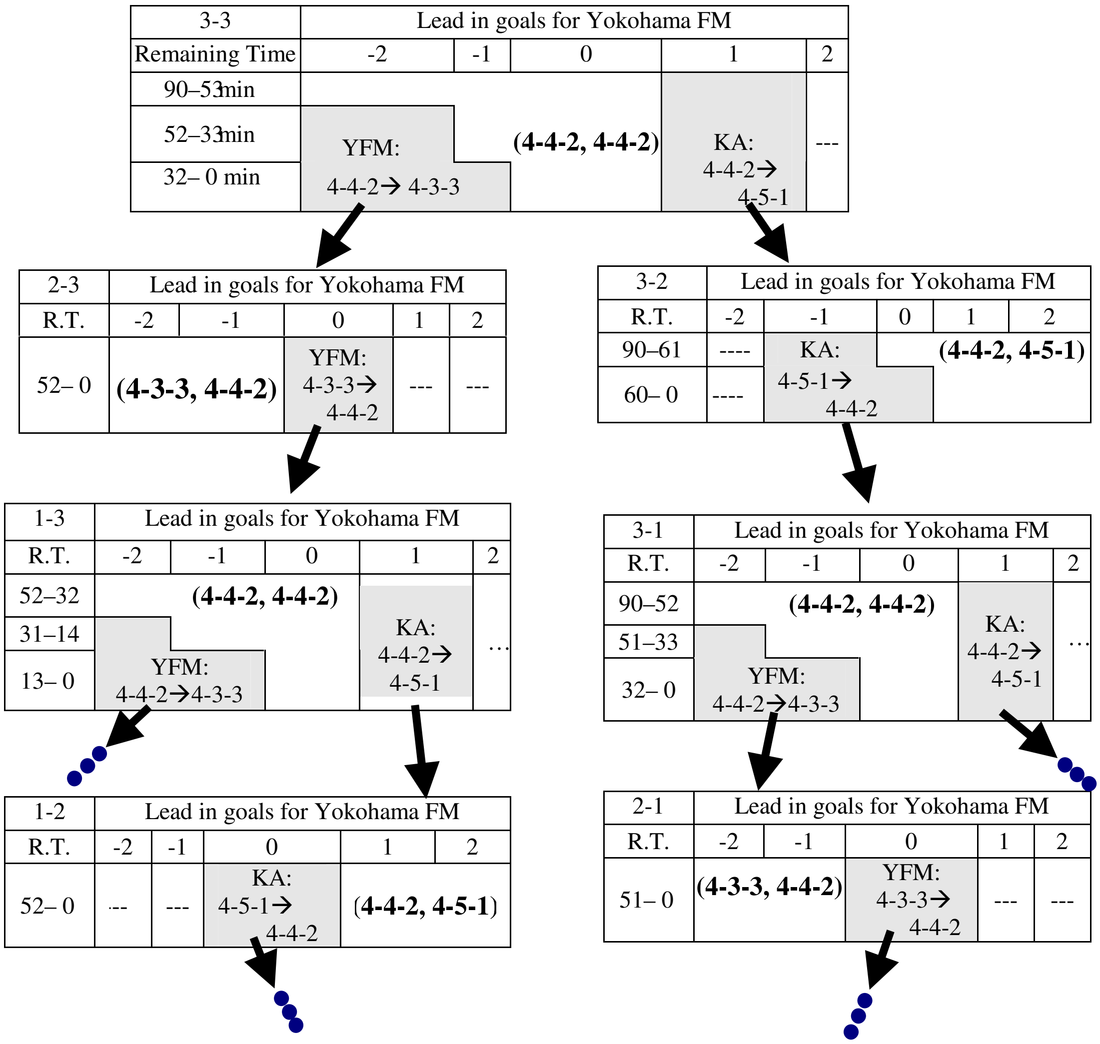


Figure 3 Flow of the change of formation when both teams make their best decisions (YFM: Yokohama FM, KA: Kashima A , "*s-w*" such as "2-3" in the upper left cell in each block represents the number of substitutions being available for Yokohama FM and Kashima A, respectively.)

Here we follow the right part of the flow in Figure 3, after Kashima A has made the change from 4-4-2 to 4-5-1 in the top block (3-3). If it levels the scores after going behind, and there are less than 60 minutes remaining, it should change its formation back from 4-5-1 to 4-4-2, as shown in block (3-2), after which the formations of both teams are 4-4-2 again (block (3-1)). In this situation Yokohama FM still has 3 possible substitutions available. If Yokohama FM takes the lead again, Kashima A should once again make a substitution to change its formation from 4-4-2 to 4-5-1.

Otherwise, if Yokohama FM falls behind, in the situation shown in block (3-1), Yokohama FM should make a substitution to change its formation from 4-4-2 to 4-3-3. This substitution leads to the situation shown in block (2-1). After this situation, if Yokohama FM subsequently levels the scores, it should make a substitution to change the formation back from 4-3-3 to 4-4-2.

Although we do not show the other part of the flow in Figure 3, the situation is much the same. If either team is ahead or level, it should adopt a 4-4-2 formation; if Kashima A falls behind it should use 4-5-1; and if Yokohama FM falls behind, it should use 4-3-3.

4.4. The effect of the introduction of the tactical change of the opposing team

We now look at the effect of the introduction of the tactical change of the opposing team on the optimal timing for substitutions.

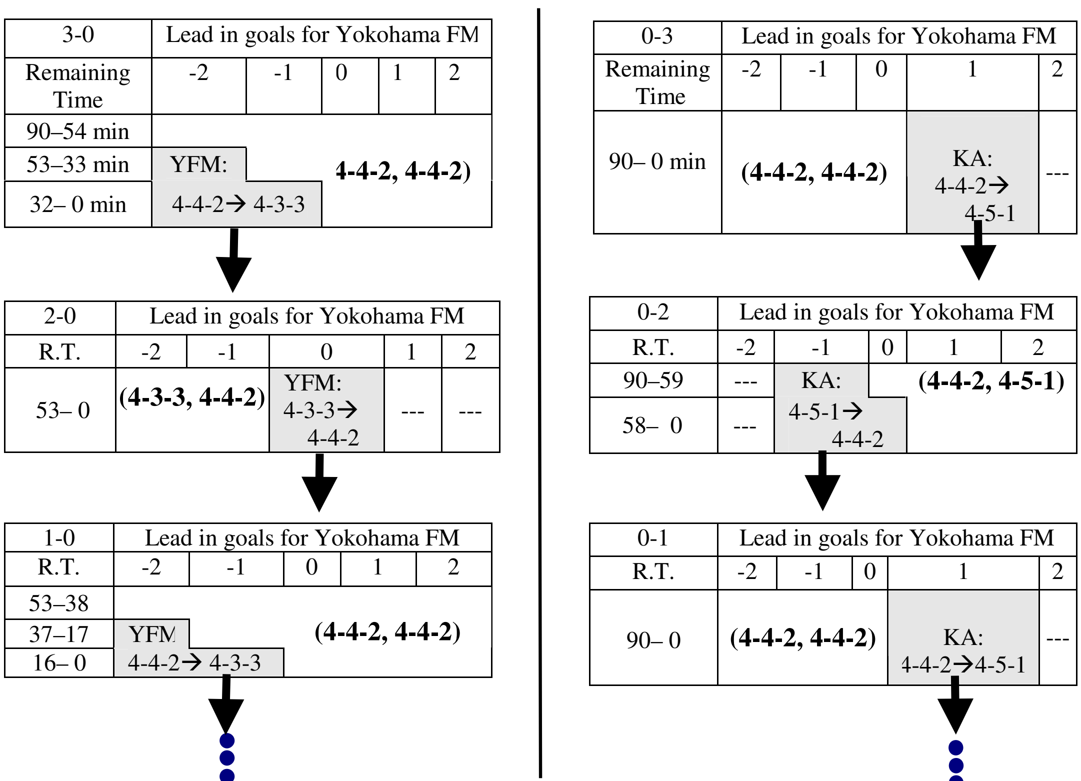
Firstly, we compare between the above result and the result of the case where only one team has 3 substitutes being available, starting off with the same formations (4-4-2, 4-4-2). Figure 4 (a) and (b) shows the flows of the change of the best formations when only Yokohama FM has 3 substitutions being available and only Kashima A has them, respectively. By comparing between blocks (3-3) (2-3) (1-3) in Figure 3 and blocks (3-0) (2-0) (1-0) in Figure 4(a), the pattern of the change of the formations is almost the same for Yokohama FM, that is, if Yokohama FM falls behind by 1 or 2 goals, it should make a substitution to change its formation from 4-4-2 to 4-3-3, and then if it gets back from behind to be level, it should return its formation from 4-3-3 to 4-4-2.

However, the timing of the change is a little different, comparing block (1-3) of Figure 3 and block (1-0) of Figure 4(a). If Kashima A is allowed to change its formation, the timing of the substitution of Yokohama FM should be delayed. For example, in the case where Yokohama FM is behind by one goal, Yokohama FM should make a substitution to change its formation from 4-4-2 to 4-3-3 when there are 16 minutes remaining in the case where Kashima A has no substitutions available, as shown in block (1-0) of Figure 4(a), but with 13 minutes remaining if Kashima A has 3 substitutions available as shown in block (1-3) of Figure 3. That is, there is a 3 minute delay in the case where the opposing team's tactical change is considered.

This delay seems reasonable because there is a possibility for Yokohama FM to score 2 goals (i.e. turn the tables) in this 3 minutes, which will result in Kashima A changing its formation from 4-4-2 to 4-5-1 which is more offensive. However, if Kashima A is not allowed to make a substitution, Kashima A can not use the offensive formation (4-5-1). Thus, the possibility that Kashima A scores a goal after Yokohama FM turns the tables is decreased by not allowing Kashima A to make a substitution. According to our calculation, the probability of Yokohama FM winning with 16 minutes remaining in the case where Kashima A has no substitution available is 0.1113, although this value decreases to 0.1098 in the case where Kashima A has 3 substitutions available. Although the difference between these values is quite small, Yokohama FM has some benefit if Kashima A has no substitutions available, and can make a substitution to change the formation from 4-4-2 to 4-3-3 three minutes earlier without being afraid that Kashima A will use its offensive formation (4-5-1).

We also compare between block (3-2) in Figure 3 and block (0-2) in Figure 4(b), and obtain similar implications for Kashima A, that is, if the scores are level or Kashima A leads (i.e. Yokohama FM is behind) by one goal, it should make a substitution to change its formation from 4-5-1 to 4-4-2, and then if Kashima A falls behind (i.e. Yokohama FM leads), Kashima A should return its formation from 4-4-2 to 4-5-1. The timing of the change is also a little different between them, as shown in block (3-2) of Figure 3 and block (0-2) of Figure 4(b). In this case, if Yokohama FM is allowed to change its formation, the timing of the substitution of Kashima A is at 60 minutes remaining, which is 2 minutes earlier than the case when Yokohama FM is not able to make a substitution.

Here we infer that if the scores are level or Kashima A leads (i.e. Yokohama FM is behind), Kashima should change its formation from 4-5-1 to 4-4-2 which is more defensive. If Yokohama FM has 3 substitutions available, Yokohama FM will change its formation from 4-4-2 to 4-3-3 which is more offensive in order to try to level the scores, and it will return its formation from 4-3-3 to 4-4-2 if and when it does so. Thus it is better for Kashima A to be defensive earlier in the case where Yokohama FM is allowed to make a substitution.



(a) 3 substitutions available only for Yokohama FM

(b) 3 substitutions available only for Kashima A

Figure 4 Flow of the change of formation when both teams make their best decisions (YFM: Yokohama FM, KA: Kashima A , "*s-w*" such as "2-3" in the upper-left cell in each table represents the number of substitutions being available for Yokohama FM and Kashima A, respectively.)

Secondly, we calculate the probability of Yokohama FM winning depending on the number of substitutions being available for both team. Although we just show a situation of starting formation (4-3-3, 4-4-2) in Table 6, the effect of whether there are available substitutions or not is quite large. For example, if Kashima A has even one substitution available, its probability of winning is around 0.02 – 0.05 percentage points higher compared to the case of no substitutions being available. Similarly, if Yokohama FM has even one substitution available, it has a benefit of around 0.1 - 0.2 percentage points in probability compared to no substitutions being available.

Table 6 Probability of Yokohama FM winning the match at home based on the number of substitutions available, starting formations 4-3-3 of Yokohama FM and 4-4-2 of Kashima A.

$P^{sw}(0,90,f,g)$		*w* (Number of substitutions available for Kashima A)			
		3	2	1	0
s (Number of substitutions available for Yokohama FM)	3	0.734	0.735	0.736	0.752
	2	0.734	0.734	0.736	0.751
	1	0.733	0.734	0.735	0.751
	0	0.591	0.594	0.605	0.660

4.5. The Effect of the quality of decisions on the probability of winning

In order to estimate the effect of the quality of decisions of the formation change on the probability of winning the match, we study the other combinations of managers' decision, i.e. the cases where one manager always takes the best, another takes the worst, and vice versa, or both managers always take their worst decisions. To analyse these combination of decisions, we take "$\max_i \max_j$", "$\min_i \min_j$" and "$\min_i \max_j$" in expression (6) instead of "$\max_i \min_j$", respectively.

Firstly, if Yokohama FM is assumed to always take its best decision of the formation changes and Kashima A always takes its worst, the probability of Yokohama FM winning the march is calculated by taking "$\max_i \max_j$" in expression (6), and we get 0.773 by starting with formations 4-4-2 of Yokohama FM and 4-5-1 of Kashima A. Conversely, if Yokohama FM is assumed to always take its worst decision and Kashima A always takes its best, the probability of Yokohama FM winning the march is calculated by taking "$\min_i \min_j$" in expression (6), and we get 0.57889 by starting with formations 4-3-3 of Yokohama FM and 4-5-1 of Kashima A.

If both teams always choose their worst tactics, they start with formation 3-5-2 for Yokohama FM and 4-4-2 for Kashima A and the probability of Yokohama FM winning becomes 0.657. These results are summarised in Table 7.

Table 7 Probability of Yokohama FM winning the match at home in different combination of the tactics

Tactics		Decision rule	Probability of Yokohama FM winning	Starting formation	
Yokohama FM	Kashima A			Yokohama FM	Kashima A
Best	Worst	Max-max	0.773	4-4-2	4-5-1
Best	Best	Max-min	0.734	4-4-2	4-4-2
Worst	Worst	Min-man	0.657	3-5-2	4-4-2
Worst	Best	Min-min	0.579	4-3-3	4-5-1

These results are, of course, specific only to matches between Yokohama FM and Kashima A based on the data of the 2002 season, and for other teams, or against different opponents, or even in different seasons, the tactical changes of formation by substitutions with the choice of starting formation, could be different. However, these results give an estimate of the change in the probability of winning in relation to the quality of decision making by managers, and this method could be extended as a tool to evaluate quantitatively the tactical decisions of a manager during a match.

We also study the tactical changes of formation in Yokohama FM's away match. The probabilities of Yokohama FM winning the match (starting with each formation) is tabulated in Table 8. The probabilities of Yokohama FM winning away are less than those at home as expected, and the formation changes are not same as those at home.

Table 8 Probability of Yokohama FM winning the match away in different combination of the tactics.

Tactics		Decision rule	Probability of Yokohama FM winning	Starting formation	
Yokohama FM	Kashima A			Yokohama FM	Kashima A
Best	Worst	Max-max	0.670	4-4-2	4-4-2
Best	Best	Max-min	0.621	4-4-2	4-5-1
Worst	Worst	Min-max	0.531	4-3-3	4-4-2
Worst	Best	Min-min	0.414	4-3-3	4-5-1

4.6. The actual matches between Yokohama FM and Kashima A

It is interesting to compare these results with the formation changes actually adopted by Yokohama FM and Kashima A in their two matches during the 2002 season.

In Yokohama FM's home match, Yokohama FM and Kashima A adopted formations 4-3-3 and 4-5-1 respectively for the first half, although Table 5 suggests that each team should have started with formation 4-4-2. During the first half,Yokohama FM scored two goals without conceding any goals, so its tactic could be said to have been successful. Formation 4-5-1 for Kashima is the best when it is behind. So the strategy actually adopted accords in part, but only in part, with the tactic suggested by our model. In the second half, Kashima A changed to formation 4-4-2, which is not recommended by our model, but there were no further goals. With thirteen minutes remaining, Yokohama FM changed to formation 4-4-2, which is recommended by the model, and used uncommon formation 5-3-2 for in the last minute. The result was that Yokohama FM won the match by two goals to zero.

In Yokohama FM's away match, Yokohama FM and Kashima A started with formation 3-5-2 and 4-4-2 respectively, although our model recommends 4-4-2 and 4-5-1 respectively. This formation was kept for the first 71 minutes. In this match, Kashima A led by one goal after 53 minutes, but Yokohama FM levelled the score after 70 minutes. One minute later, Kashima A changed to formation 4-5-1, as our model suggests, then 4 minutes later Kashima A scored another goal. Three minutes later Yokohama FM changed to an uncommon formation 3-4-3, and three minutes later, Kashima A also changed to an uncommon formation 5-4-1 and these formations were kept to the end of the match. Since these two formations are not included in our analysis because of shortage of data, we cannot comment on the wisdom or otherwise of these changes. The end result was that no further goals were scored and Kashima A eventually won the match by two goals to one.

Anyway, it is of course very unwise to reach even tentative conclusions on the basis of only two matches. However, if we were to conduct the same analysis for each team of the league over a full season or more, we could evaluate the teams' decisions of formation changes by comparing the suggestions from game theory with the real decisions of managers.

5. Reversible tactics

Until now, we discussed the tactical change of formation made by substitutions, which are not reversible, that is, once making a substitution, the player who left the match is not allowed to return. However, although substitutions are not reversible, it may be possible to achieve a similar effect by other means. For example, a midfielder may be instructed to play as a defender or as an attacker, and this change of formation is reversible. The midfielder can then revert to his normal position after the desired effect is achieved. This section examines changes of formation as reversible tactics of this type.

Suppose that two teams have their different formations at its disposal. It is now possible for both teams to switch between its formations at will. Expression (6) is superseded by the following recursive simultaneous equations.

$$P(r,t+dt,f,g) = \max_i \min_j \begin{cases} P(r+1,t,f,g)\cdot\lambda(f,j)dt + P(r-1,t,f,g)\cdot\mu(f,g)dt + P(r,t,f,g)\cdot\{1-((\lambda(f,g)+\mu(f,g))dt\} \\ \quad :\text{Tactics i} = 0 \text{ and j} = 0 \text{ (no substitution at time t remaining)} \\ P(r+1,t,f',g)\cdot\lambda(f',g)dt + P(r-1,t,f',g)\cdot\mu(f',g)dt + P(r,t,f',g)\cdot\{1-(\lambda(f',g)+\mu(f',g))dt\} \\ \quad :\text{Tactics i} = 1 \text{ and j} = 0 \text{ (now making the substitution } (f\to f') \text{ at time t remaining)} \\ P(r+1,t,f,g')\cdot\lambda(f,g')dt + P(r-1,t,f,g')\cdot\mu(f,g')dt + P(r,t,f,g')\cdot\{1-(\lambda(f,g')+\mu(f,g'))dt\} \\ \quad :\text{Tactics i} = 0 \text{ and j} = 1 \text{ (now making the substitution } (g\to g') \text{ at time t remaining)} \\ P(r+1,t,f',g')\cdot\lambda(f',g')dt + P(r-1,t,f',g')\cdot\mu(f',g')dt + P(r,t,f',g')\cdot\{1-(\lambda(f',g')+\mu(f',g'))dt\} \\ \quad :\text{Tactics i} = 1 \text{ and j} = 1 \text{ (now making the substitution } (f\to f') \text{ and } (g\to g') \text{ at time t remaining)} \\ \quad \ldots\ldots\ldots \end{cases} \tag{7}$$

where P(r,t,f,g) is the probability of a home team winning a match with time t minutes remaining when the team leads by r goals, in reversible formations f taken by the home team and g by the away team.

Table 9 summarises the results obtained with the reversible case. Comparing it with Tables 7 and 8, the probabilities of Yokohama FM winning are very close to each other. This is because these probabilities of the case of "irreversible" will asymptotically converge to the values of the case of "reversible", as the number of allowed substitutions goes to infinity. In this sense, the three "irreversible" formation changes are almost as useful to the manager as an unlimited number of reversible changes.

Table 9 Probability of Yokohama FM winning the match with "reversible" formation changes.

Tactics		Decision rule	Probability of Yokohama FM winning	
Yokohama FM	Kashima A		At home	Away
Best	Worst	Max-max	0.773	0.670
Best	Best	Max-min	0.734	0.621
Worst	Worst	Min-max	0.656	0.529
Worst	Best	Min-min	0.576	0.411

6. Further work

We have presented a zero-sum game model of a football match, which has been extended to take account of both teams' tactical change of formations. Based on the real data of the J. League of the 2002 season, we have illustrated its use for two J. League teams, considering their common formations. This has enabled us to quantify the effect of the decision making for the change of formation on the probability of winning the match with regard to the quality of managers' decisions. The results quantitatively show that the difference between the combinations of the best and worst tactics leads to a significant difference in terms of the probability of winning.

Although the examples given in this paper all depend on the particular teams, this method can be applied to other teams of other football leagues in the world, and it will help to evaluate quantitatively the quality of tactical decisions made by managers.

However, it should of course be borne in mind that the whole analysis is based on the assumption that the strengths for scoring and conceding a goal of given formations or teams are constant throughout the match or the season. In reality, the strengths of a formation may depend on the skill or the tiredness of players. If appropriate data can be obtained, this would enable us to test the significance of these effects and thus improve our models. We could also study the effects of a change of strength on the outcomes by using sensitivity analysis or another method.

In our study we have not dealt with the case where the game is solved using mixed strategies, in which teams select their tactics according to probabilities. This fact is related to our formulation for evaluating the offensive and defensive strengths following expressions (1) and (2). That is, these strengths are actually calculated by averaging over all the opposing teams, and we do not take into account the preference of teams by introducing any interaction terms between teams or formations. If we were to introduce another parameter which represents the specific strength of team A against team B in expressions (1) and (2), the game would then be solved as a mixed strategy, and we intend to do this in future work.

Furthermore, we have discussed the tactics as a zero-sum game using the probability of winning, but alternatively we could make the formulation as a non-zero sum game using the expected number of points gained in a match, since a won match produces three points (all for the victor) whereas a drawn match produces only two (one for each team) This will make the problem more complicated as we may have to introduce the effect of co-operation between managers.

Our model is still preliminary and longer observational studies will be required for a complete analysis, but we believe that our method proposes a way to cater for the complex problem in football based on game theory as a first step, and will lead the way towards the development of a more useful match analysis.

Appendix: Zero-sum game and saddle point

As two teams play a football match, this type of zero-sum game can be represented by such a matrix as Table A1. Each entry is a probability of the home team winning. Thus, for example, this probability is 0.7 if the home team chooses its Tactic 0 and the away team chooses its Tactic 1. In this case, the away team would have a probability of winning equal to 1 – 0.7 = 0.3, because the sum of both team's probabilities of winning the match is 1. Here, even if the sum of probabilities is not zero, both teams can still be in total conflict. So, in general, a "zero-sum game" includes this type of "constant-sum game".

Table A1 Example of zero-sum game with a saddle point

		Away team	
		Tactic 0	Tactic 1
Home team	Tactic 0	0.6	0.7
	Tactic 1	0.4	0.5

In the case of Table A1, if the home team chooses its Tactic 0, the away team will choose its Tactic 0 and hold the home team to a probability of 0.6 (the smallest number in row 1 of the matrix). Similarly, if the home team chooses Tactic 1, the away team will also choose its Tactic 0 and hold the home team to a probability of 0.4 (the smallest number in row 2 of the matrix). Thus, by choosing its Tactic 0 the home team can ensure that it will have at least 0.6 which is the largest minimum of these rows.

From the away team's viewpoint, the away team chooses its tactic and the home team will choose its tactic that makes the probability of the away team winning as small as possible (i.e. the probability of the home team as large as possible). In this case, the away team can ensure that it will reduce the home team's probability of winning to 0.6, which is the smallest maximum of these columns.

Thus, the only rational outcome is for the home team to have 0.6 and the home team cannot expect to get more than 0.6, because the away team can hold the probability of the home team winning to 0.6 by choosing Tactic 0. Here, the matrix we have just analysed has the property of satisfying max (row minimum) = min (column maximum).

If this type of zero-sum game satisfies this condition (i.e. the largest minimum of the rows equals to the smallest maximum of these columns), it is said to have a saddle point and this value is called the value of the game. This example has a saddle point and the value of the game is 0.6. A saddle point can also be thought of as an equilibrium point, in the sense that if one team were to change from the optimal tactic, it will decrease the probability of its winning. If there are not any saddle points, the game is solved as mixed strategies, in which each team selects its tactics with a probability.

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